Calculus Challenge #13 SOLUTION

Crossing a Street Using Calculus

If cars are moving along a one-lane street at a rate of 12 cars per minute, and it takes you 10 seconds to walk across the street, how long should you expect to wait before an interval between cars is large enough to allow you to cross the street? Problems like this belong to an area of mathematics known as queuing theory and use probabilistic models to describe daily events. Calculus plays an important role in queuing models, as we will see in this Challenge Problem. We will develop a general model using the intensity parameter λ and crossing time T, and then use the results to answer the question originally posed with $\lambda = 12$ and $T = 0.167$ (10 seconds). Notice that we need to be sure we are making all measurements in the same units, in this case minutes.

When we say that cars are moving along the road at a rate of 12 cars per minute, we don't really mean that a car arrives every 5 seconds. The measure 12 cars per minute is an average rate. We need a way to describe when we expect the individual cars to appear at our position on the side of the street. After watching traffic flow for many years, mathematicians have decided that the exponential distribution function $F(t) = \lambda e^{-\lambda t}$ is a good way to describe the interarrival time between cars. The interarrival times (the time between cars) determines if there is enough time for you to cross the street. In free flowing traffic, the interarrival times are assumed to be independent. The function $F(t) = \lambda e^{-\lambda t}$, with $t > 0$, is a called a probability density (or distribution) function with intensity λ. A probability density function has two essential characteristics:

1) $F(t) \geq 0$ for all t in its domain.

2) The total area under the curve over the domain of the function is 1. This means $\int_0^\infty \lambda e^{-\lambda t}\,dt = 1$.

If we think of time 0 as the time a car has just passed your position, the probability that the next car arrives in the time interval $[t_0, t_1]$ is given by the area under the graph of F from t_0 to t_1. That is,

$$P(\text{Next car arriving between } t_0 \text{ and } t_1) = \int_{t_0}^{t_1} \lambda e^{-\lambda t}\,dt\ .$$

Figure 1 illustrates the problem setting. We will define gap G_k to be the gap between cars $k-1$ and k. Time $t = 0$ denotes the time when Car 0 has just passed your position. The cars are arriving at rate λ cars per minute with interarrival times distributed exponentially, and it takes T minutes for you to safely walk across the street. The cars are moving right to left in Figure 1.

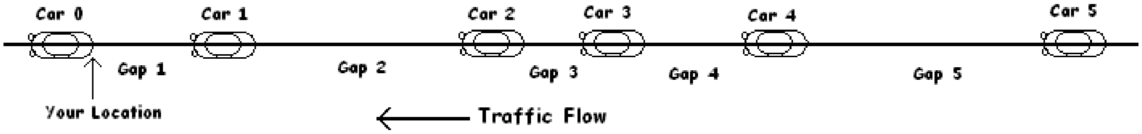

Figure 1: Diagram of cars and gaps

1. Find the probability that Car 1 arrives before time T (this is also the probability that Gap 1 is too short for you to cross in time).

The probability that Car 1 arrives before time T is $\int_0^T \lambda e^{-\lambda t}\,dt = 1 - e^{-\lambda T}$. This is the same for all cars. After a car has passed, the next car can be thought of as Car 1.

2. Find the probability that Car 1 does not arrive before time T (this is also the probability that Gap 1 is long enough for you to cross in time).

Since the probability Car 1 arrives before time T is $1-e^{-\lambda T}$, the probability that Car 1 does not arrive before time T is $1-\left(1-e^{-\lambda T}\right)=e^{-\lambda T}$. This is also the same for all cars.

3. Using the assumption that the interarrival times are independent, explain why the probability that Gap 4 is the first gap big enough for you to cross (first gap greater than T) the street is given by $P\left(G_4\right)=\left(1-e^{-\lambda T}\right)^3\left(e^{-\lambda T}\right)$.

If Gap 4 is the first gap big enough for you to cross, then Gaps 1, 2, and 3 must have been too small. If the interarrival times are independent, then the probability that Gap 1 and Gap 2 and Gap 3 are all too small is the product of the individual probabilities. So the probability that Gaps 1, 2, and 3 are too small and Gap 4 is large enough is the product of $P\left(G_4\right)=\left(1-e^{-\lambda T}\right)\cdot\left(1-e^{-\lambda T}\right)\cdot\left(1-e^{-\lambda T}\right)\cdot\left(e^{-\lambda T}\right)$. So $P\left(G_4\right)=\left(1-e^{-\lambda T}\right)^3\left(e^{-\lambda T}\right)$.

4. Using the notation $P\left(G_k\right)$ to represent the probability that the first gap with a length greater than T is Gap k, find $P\left(G_1\right)$, $P\left(G_2\right)$, $P\left(G_3\right)$, and the general probability $P\left(G_k\right)$. Also, compute $\sum_{i=1}^{\infty}P\left(G_k\right)$. Does this sum make sense?

By the same reasoning as in 3), we see that

$$P\left(G_1\right)=\left(e^{-\lambda T}\right)$$
$$P\left(G_2\right)=\left(1-e^{-\lambda T}\right)\left(e^{-\lambda T}\right)$$
$$P\left(G_3\right)=\left(1-e^{-\lambda T}\right)^2\left(e^{-\lambda T}\right)$$

and

$$P\left(G_k\right)=\left(1-e^{-\lambda T}\right)^{k-1}\left(e^{-\lambda T}\right)$$

The sum of the probabilities is $\sum_{i=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k-1}\left(e^{-\lambda T}\right)=\left(e^{-\lambda T}\right)\sum_{i=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k-1}$. Students should recognize this as an infinite geometric series with common ratio $\left(1-e^{-\lambda T}\right)$. So $\left(1+\left(1-e^{-\lambda T}\right)+\left(1-e^{-\lambda T}\right)^2+\left(1-e^{-\lambda T}\right)^3+\cdots\right)=\dfrac{1}{1-\left(1-e^{-\lambda T}\right)}=\dfrac{1}{e^{-\lambda T}}$. Finally, the sum is $\left(e^{-\lambda T}\right)\sum_{i=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k-1}=\dfrac{e^{-\lambda T}}{e^{-\lambda T}}=1$. We certainly expect the total probability to be 1, so this result is not a surprise.

The distribution of probabilities $P(G_k)$ for $k=1,2,3,\ldots$ is known as a geometric distribution. It is a discrete probability distribution, since the domain is the positive integers. We want to know, on average, how many gaps we will need to look at before we find one long enough for us to cross the street. We need to find the expected value of *k*, denoted $\overline{k}$. The expected value of a discrete probability distribution is found by computing the infinite sum $\overline{k}=\sum_{i=1}^{\infty} k\cdot P(G_k)$. In this case, we need to compute the sum of $\sum_{i=1}^{\infty} k\cdot \overbrace{\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}}^{P(G_k)}$. Notice that this infinite series is not geometric, so we can't use our formula for the sum of an infinite geometric series.

5. To find the sum $\overline{k}=\sum_{i=k}^{\infty} k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$, first notice that the summand $k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$ looks very much like the derivative with respect to *T* of some function. Write the sum $\overline{k}=\sum_{i=k}^{\infty} k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$ as a constant times the sum of a derivative.

The summand $k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$ is almost $\frac{d}{dT}\left(1-e^{-\lambda T}\right)^{k}$. Since $\frac{d}{dT}\left(1-e^{-\lambda T}\right)^{k}=k\left(1-e^{-\lambda T}\right)^{k-1}\left(-e^{-\lambda T}\right)(-\lambda)$, we see that the summand is $\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^{k}\right]$.
So, $\overline{k}=\sum_{k=1}^{\infty} k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}=\left(\frac{1}{\lambda}\right)\sum_{k=1}^{\infty}\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^{k}\right]$.

7. We know that for finite sums, the sum of derivatives is the derivative of a sum. We won't prove that this infinite sum also obeys this rule, but it does. Rewrite the sum of derivatives as the derivative of a sum.

If the sum of the derivatives is equal to the derivative of the sum, then $\left(\frac{1}{\lambda}\right)\sum_{k=1}^{\infty}\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^{k}\right]=\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k}\right]$. Now the sum is of an infinite geometric series, so we know how to compute it.

8. You should recognize that the sum is now an infinite geometric series. Find the sum, and then differentiate the result. Show that you would expect to wait for $e^{\lambda T}-1$ cars to pass before crossing.

Since $\left(\frac{1}{\lambda}\right)\sum_{k=1}^{\infty}\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^{k}\right]=\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k}\right]$, we know the sum in this expression is

$\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k} = \frac{1-e^{-\lambda T}}{1-\left(1-e^{-\lambda T}\right)} = \frac{1-e^{-\lambda T}}{e^{-\lambda T}} = e^{\lambda T}-1$. So

$\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k}\right] = \left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\left(e^{\lambda T}-1\right)\right] = \left(\frac{1}{\lambda}\right)\left(\lambda e^{\lambda T}\right) = e^{\lambda T}$, as required.

9. On average, how many cars will you watch go by on our street with $\lambda = 12$ and $T = 0.167$?

a) How long would you expect this to take?

If $\lambda = 12$ and $T = 0.167$, then we would expect to wait for $e^{12(0.167)} - 1 = e^{2} - 1 \approx 6.4$ cars to pass. If cars arrive at a rate of one every 5 seconds, we would have to wait, on average, 32 seconds before we found a gap between cars large enough for us to cross.

b) How long would you expect to wait if you were on crutches and it took 30 seconds for you to cross the street?

If you were on crutches and it took 30 seconds ($T = 0.5$), then you would expect to wait for $e^{12(0.5)} - 1 = e^{6} - 1 \approx 402$ cars to pass. You should begin looking for an alternate route now.

c) If you dashed across in 6 seconds?

If you could dash across in 6 seconds, then you would expect to wait for only $e^{12(0.1)} - 1 = e^{1.2} - 1 \approx 2.3$ two or three cars before crossing.

10. In many places, if the expected wait is longer than 90 seconds, a crossing light may be installed to facilitate crossing. If $T = 0.167$, what must λ be for a crossing light to be needed?

If a crossing light is needed whenever the expected wait is longer than 90 seconds, for $T = 0.167$, the traffic intensity that would necessitate a crossing light can be found by solving $\left(e^{\lambda(0.167)} - 1\right)\left(\frac{1}{\lambda}\right) = 1.5$. This equation cannot be solve analytically, so we can use Newton's Method or another numerical technique to find $\lambda \approx 20.78$ as the solution.

References:

Mooney, Douglas, and Randall Swift, *A Course in Mathematical Modeling*, Mathematical Association of America, 1999.

Mesterton-Gibbons, Michael, *A Concrete Approach to Mathematical Modeling*, Addison-Wesley Publishing Company, Redwood City, CA, 1989.