NCAAPMT Calculus Challenge 2010-2011

Challenge #2 Due: October 13, 2010

An Alternate Form of the Derivative

In class, you have learned the definition of the derivative as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists, and you used this limit definition to find derivative rules for a variety of functions. This definition is often called the forward difference definition, since you start at x and move forward a distance *h*.

There are other possible definitions of derivative that could be used. One is known as the symmetric difference. In this definition, you have

$$f_s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}, \text{ provided the limit exists.}$$

This definition is often used when an approximation of the derivative is needed. In this challenge, you will see both the strengths and weaknesses of this definition.

a) Most calculators compute the symmetric difference $\frac{f(x+h) - f(x-h)}{2h}$ with a small value of *h* when it finds a numerical derivative. You can find the value of *h* that your calculator uses by comparing the value of the computed numerical derivative of e^x at $x = 0$ to the values of $\frac{e^{(0+h)} - e^{(0-h)}}{2h}$ for various values of *h*. What calculator do you use and what is its default value of *h*? Be sure to show how you determined the size of *h*.

Pulling out my TI-83, I first find nDeriv(e^(x),x,0) = 1.000000167 (the syntax of your calculator may differ). I want to match this value with values of $\frac{e^{(h)} - e^{(-h)}}{2h}$.

We see that the match comes from using $h = 0.001$. This is the default for the TI-83. Notice that if we decrease the size of h by a factor of 10, our error decreases by a factor of 100. Newton's method is said to converge quadratically.

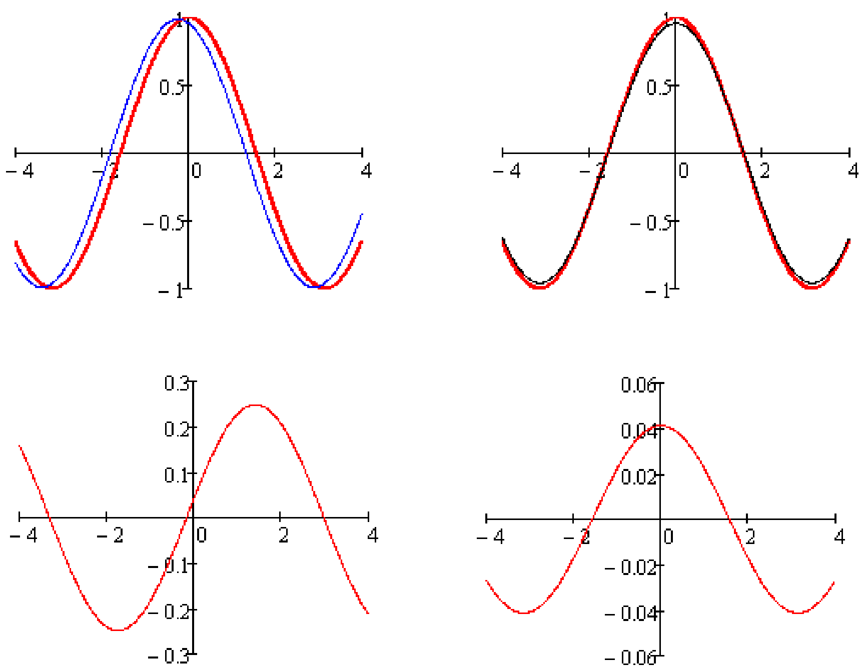
h	$\frac{e^{(h)} - e^{(-h)}}{2h}$
0.1	1.0016675
0.01	1.000016667
0.001	**1.000000167**
0.001	1.00000000167
0.0001	1.0000000000167

b) How much better is the symmetric difference, $\frac{f(x+h) - f(x-h)}{2h}$, than the classical difference quotient, $\frac{f(x+h) - f(x)}{h}$, for approximating derivatives? Use $f(x) = \sin(x)$ and

$h = 0.01$. Compare the graphs of $y = \dfrac{f(x+h) - f(x-h)}{2h}$ and $y = \dfrac{f(x+h) - f(x)}{h}$ to the graph of the cosine function to see the errors in each derivative approximation. Which appears to have the smaller errors? From your graphs, for what values of x does the error in each appear to be the greatest? Can you explain why the largest error happens there?

In the plots below, the forward and symmetric differences are graphed against the cosine curve. The value of h is large (0.5), so it is easier to see the error. The cosine curve is in red and the forward approximation in blue on the left. The symmetric approximation is in black on the right.

$$D(x) := \frac{f(x+h) - f(x)}{h} \qquad S(x) := \frac{f(x+h) - f(x-h)}{2 \cdot h}$$



Error for Forward Difference　　Error for Symmetric Difference

Notice that the error in the forward difference amounts to a horizontal shift, with the maximum error at the point of inflection, while the error in the symmetric difference is a vertical compression, resulting in a maximum error at the extreme values.

c) Use the limit definition of the symmetric difference, $f_s'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$, to derive the derivative formulas for x^2, $\dfrac{1}{x}$, and $\sin(x)$ at $x = a$. You should find using this definition particularly helpful with the sine function.

For $f(x) = x^2$, we have $f_s'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 - (x-h)^2}{2h}$, so

$$\lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2\right) - \left(x^2 - 2xh + h^2\right)}{2h} = \lim_{h \to 0} \frac{4xh}{2h} = 2x.$$

For $f(x)=\frac{1}{x}$, we have $f_s'(x)=\lim_{h\to 0}\frac{\frac{1}{(x+h)}-\frac{1}{(x-h)}}{2h}$, so

$$\lim_{h\to 0} f_s'(x)=\lim_{h\to 0}\frac{\frac{(x-h)-(x+h)}{(x+h)(x-h)}}{2h}=\lim_{h\to 0}\frac{\frac{-2h}{(x+h)(x-h)}}{2h}=-\frac{1}{x^2}.$$

For $f(x)=\sin(x)$, we have $f_s'(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x-h)}{2h}$, so

$$f_s'(x)=\lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-(\sin(x)\cos(h)-\cos(x)\sin(h))}{2h}=$$

$$\lim_{h\to 0}\frac{2\cos(x)\sin(h)}{2h}=\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}=\cos(x).$$

We still needed $\lim_{h\to 0}\frac{\sin(h)}{h}=1$, but we didn't need to do any convoluted substitutions.

d) The position of a falling body (parachutist) has been measured with the corresponding time since the parachutist began her free-fall. The data appear below. Approximate the velocity and acceleration using the classical difference quotient and with the symmetric difference quotient. Note that you cannot find estimates for all of the times and that you will have one fewer estimate with the symmetric difference.

Data	*Time* (secs)	2	4	6	8	10	12
	Position (ft)	63.1	238.2	493.8	798.5	1130.3	1476.2
Classical Difference	*Velocity* (m/sec)	87.55	127.80	152.35	165.9	172.95	**
	Acceleration (ft/sec ^{2})	35.125	12.275	6.775	3.525	**	
Symmetric Difference	*Velocity* (ft/sec)	**	107.675	140.075	159.125	169.425	**
	Acceleration (ft/sec ^{2})	**	**	12.8625	7.3375	**	**
True Values	*Velocity* (m/sec)	61.8	110.5	142.3	160.5	170.2	175.1
	Acceleration (ft/sec ^{2})	28.4	20.1	12.1	6.6	3.4	1.7

Use the symmetric difference to estimate the velocity and acceleration of the parachutist. Why are the velocity estimates higher than the true values?

The position function is increasing and concave down, so our tangent line estimates are always larger than the true values.

So far, the symmetric difference seems to be winning each challenge with the classical difference quotient. Why don't we use it as the standard definition?

e) What happens if we use the symmetric derivative to find the derivative at $x = 0$ for $|x|$?

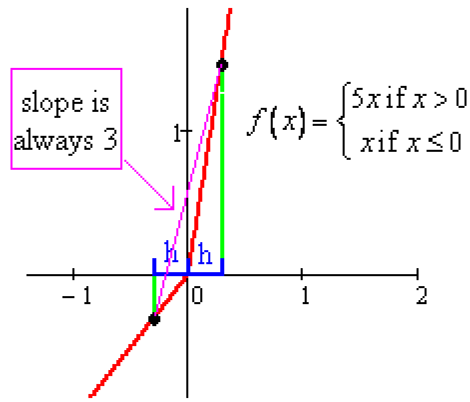
For $f(x) = |x|$, we have $f_s'(0) = \lim_{h\to0}\frac{|0+h|-|0-h|}{2h}$, so $f_s'(0) = \lim_{h\to0}\frac{|0+h|-|0-h|}{2h}\lim_{h\to0}\frac{|h|-|-h|}{2h} = 0$. If we use the symmetric derivative as our definition, then we must allow the absolute value function to be differentiable at $x = 0$.

f) Does $f(x) = \begin{cases} 5x \text{ if } x > 0 \\ x \text{ if } x \le 0 \end{cases}$ have a symmetric derivative at $x = 0$? Draw a graph to illustrate how the slope of the tangent is being computed and why it is a problem.

For $f(x) = \begin{cases} 5x \text{ if } x > 0 \\ x \text{ if } x \le 0 \end{cases}$, we have $f_s'(0) = \lim_{h\to0}\frac{5(0+h)-(0-h)}{2h}$.

So, $f_s'(0) = \lim_{h\to0}\frac{6h}{2h} = 3$.

Again, we have a derivative at a corner point. This is one reason we don't choose this definition for derivative.



Questions d) and e) show the fatal flaw in the symmetric difference as a definition of derivative. It remains, however, the choice for approximating the value of a derivative at a point.

g) Use several simple functions for *f* to explore the limit, $\lim_{h\to0}\frac{f(x-2h)-2f(x)+f(x+2h)}{4h^2}$. This limit is based on the symmetric difference. What is being computed with this limit?

Let's consider $f(x) = x^3$ and $f(x) = \sin(x)$.

For $f(x) = x^3$, we have

$$\lim_{h\to0}\frac{(x-2h)^3-2(x)^3+(x+2h)^3}{4h^2} = \lim_{h\to0}\frac{\left(x^3-6x^2h+12xh^2-h^3\right)^3-2x^3+\left(x^3+6x^2h+12xh^2+h^3\right)}{4h^2}.$$

Combining terms we see that $\lim_{h\to0}\frac{24xh^2}{4h^2} = 6x$. This is the second derivative of $f(x) = x^3$.

For $f(x) = \sin(x)$, we have

$$\lim_{h\to 0}\frac{\sin(x-2h)-2\sin(x)+\sin(x+2h)}{4h^2}$$
$$=\lim_{h\to 0}\frac{\sin(x)\cos(2h)-\cos(x)\sin(2h)-2\sin(x)+\sin(x)\cos(2h)+\cos(x)\sin(2h)}{4h^2}.$$

Combining terms, we have
$$\lim_{h\to 0}\frac{2\sin(x)\cos(2h)-2\sin(x)}{4h^2}=\lim_{h\to 0}\frac{\sin(x)(\cos(2h)-1)}{2h^2}=\sin(x)\left(\lim_{h\to 0}\frac{(\cos(2h)-1)}{2h^2}\right).$$

Using the double angle identity, we have

$\lim_{h\to 0}\frac{\cos(2h)-1}{2h^2}=\lim_{h\to 0}\frac{(1-2\sin^2(h))-1}{2h^2}=\left(\frac{-2}{2}\right)\lim_{h\to 0}\frac{\sin(h)}{h}\cdot\frac{\sin(h)}{h}=-1$, so
$\sin(x)\left(\lim_{h\to 0}\frac{(\cos(2h)-1)}{2h^2}\right)=-\sin(x)$. Again, this is the second derivative of $f(x)=\sin(x)$.

Will this always be the case?

If $f_s'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$ then
$$f_s''(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=\lim_{h\to 0}\frac{f((x+h)+h)-f((x+h)-h)-\left(f((x-h)+h)-f((x-h)-h)\right)}{2h}.$$
Combining terms we see that

$\lim_{h\to 0}\frac{f(x+2h)-f(x)-f(x)+f(x-2h)}{2h}=\lim_{h\to 0}\frac{f(x+2h)-2f(x)+f(x-2h)}{2h}$ is indeed the second derivative.