NCAAPMT Calculus Challenge 2010-2011

Challenge #2 Due: October 13, 2010

An Alternate Form of the Derivative

In class, you have learned the definition of the derivative as $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists, and you used this limit definition to find derivative rules for a variety of functions. This definition is often called the forward difference definition, since you start at x and move forward a distance *h*.

There are other possible definitions of derivative that could be used. One is known as the symmetric difference. In this definition, you have

$$f_s'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}\text{, provided the limit exists.}$$

This definition is often used when an approximation of the derivative is needed. In this challenge, you will see both the strengths and weaknesses of this definition.

a) Most calculators compute the symmetric difference $\frac{f(x+h)-f(x-h)}{2h}$ with a small value of *h* when it finds a numerical derivative. You can find the value of *h* that your calculator uses by comparing the value of the computed numerical derivative of e^x at $x = 0$ to the values of $\frac{e^{(0+h)}-e^{(0-h)}}{2h}$ for various values of *h*. What calculator do you use and what is its default value of *h*? Be sure to show how you determined the size of *h*.

b) How much better is the symmetric difference, $\frac{f(x+h)-f(x-h)}{2h}$, than the classical difference quotient, $\frac{f(x+h)-f(x)}{h}$, for approximating derivatives? Use $f(x)=\sin(x)$ and $h = 0.01$. Compare the graphs of $y=\frac{f(x+h)-f(x-h)}{2h}$ and $y=\frac{f(x+h)-f(x)}{h}$ to the graph of the cosine function to see the errors in each derivative approximation. Which appears to have the smaller errors? From your graphs, for what values of *x* does the error in each appear to be the greatest? Can you explain why the largest error happens there?

c) Use the limit definition of the symmetric difference, $f_s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$, to derive the derivative formulas for x^2, $\frac{1}{x}$, and $\sin(x)$ at $x = a$. You should find using this definition particularly helpful with the sine function.

d) The position of a falling body (parachutist) has been measured with the corresponding time since the parachutist began her free-fall. The data appear below. Approximate the velocity and acceleration using the classical difference quotient and with the symmetric difference quotient. Note that you cannot find estimates for all of the times and that you will have one fewer estimate with the symmetric difference.

Data	*Time* (secs)	2	4	6	8	10	12
	Position (ft)	63.1	238.2	493.8	798.5	1130.3	1476.2
Classical Difference	*Velocity* (m/sec)						
	Acceleration (ft/sec ^{2})						
Symmetric Difference	*Velocity* (ft/sec)						
	Acceleration (ft/sec ^{2})						
True Values	*Velocity* (m/sec)	61.8	110.5	142.3	160.5	170.2	175.1
	Acceleration (ft/sec ^{2})	28.4	20.1	12.1	6.6	3.4	1.7

Use the symmetric difference to estimate the velocity and acceleration of the parachutist. Why are the velocity estimates higher than the true values?

So far, the symmetric difference seems to be winning each challenge with the classical difference quotient. Why don't we use it as the standard definition?

e) What happens if we use the symmetric derivative to find the derivative at $x = 0$ for $|x|$?

f) Does $f(x) = \begin{cases} 5x \text{ if } x > 0 \\ x \text{ if } x \le 0 \end{cases}$ have a symmetric derivative at $x = 0$? Draw a graph to illustrate how the slope of the tangent is being computed and why it is a problem.

Questions d) and e) show the fatal flaw in the symmetric difference as a definition of derivative. It remains, however, the choice for approximating the value of a derivative at a point.

g) Use several simple functions for *f* to explore the limit, $\lim_{h \to 0} \frac{f(x-2h) - 2f(x) + f(x+2h)}{4h^2}$. This limit is based on the symmetric difference. What is being computed with this limit?