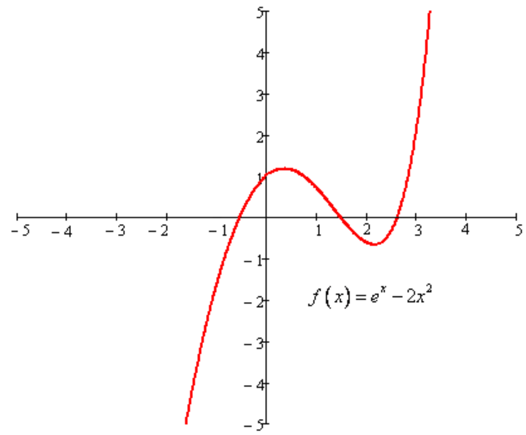
NCAAPMT Calculus Challenge 2010-2011

Calculus Challenge #3 SOLUTION

At this point in calculus, you have learned how to write the equation of a line tangent to a curve at a specific point $x = x_0$. The tangent line, $y = f'(x_0)(x - x_0) + f(x_0)$, has many important applications. In this challenge, we will use the tangent line to find approximate solutions to "unsolvable" equations. For example, suppose we want to know the values of x for which $e^x = 2x^2$. The solutions to this equation are the zeros or roots of the function $f(x) = e^x - 2x^2$. The graph of f indicates a solution between $x = -1$ and $x = 1$.



Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

should converge rapidly to the root we seek. A spreadsheet or calculator can help automate this iterative process.

If a function has more than one root, the root you find depends upon initial value x_0 you choose. Notice in our example, $x_0 = 0$ led to the solution below 0 instead of the one above 0.

First, here are some short problems to get comfortable with Newton's method:

a) Use Newton's method to find the three solutions to the equation $e^x = 2x^2$ to three decimal place accuracy.

The three zeros can be approximated using Newton's method.

$x_0 := 0$	$x_0 := 1$	$x_0 := 3$
0	1	3
-1	1.560405	2.742066
-0.626336	1.486814	2.636333
-0.544081	1.487962	2.61836
-0.539846	1.487962	2.617867
-0.539835	1.487962	2.617867
-0.539835	1.487962	2.617867
-0.539835	1.487962	2.617867
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-0.539835	1.487962	2.617867
-0.539835	1.487962	2.617867

b) There are limitations to Newton's method. What initial values of x_0 would keep the method from finding a zero for $f(x) = e^x - 2x^2$? What initial value of x_0 would lead to those values as x_1?

If the value of the derivative is zero, then the algorithm will blow up. The two critical values appear from the graph to be between 0 and 0.5 and between 2 and 2.5. We can find the zeros of the derivative by using Newton's method on the first derivative. So,

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} = x_n - \frac{e^x - 4x}{e^x - 4}$$ will generate approximations for these zeros.

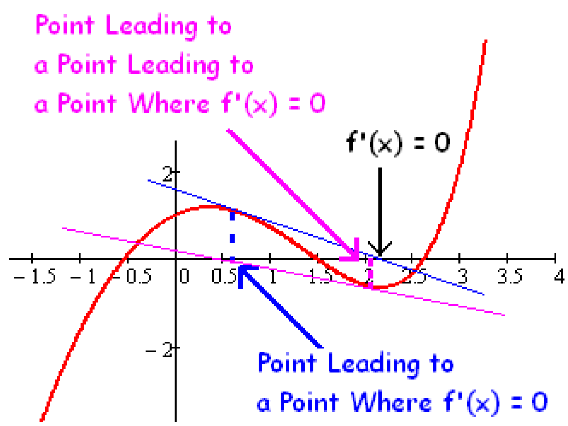
$x_0 := 0$	$x_0 := 2$
0	2
0.333333	2.18027
0.357246	2.153951
0.357403	2.153293
0.357403	2.153292
0.357403	2.153292
0.357403	2.153292
0.357403	2.153292
0.357403	2.153292
0.357403	2.153292
0.357403	2.153292

The good news is that these points (solutions to $e^x - 4x = 0$ are irrational), and since our computer will only present rational values to the algorithm, we will never have this problem.

If the derivative has rational zeros, then we need to be much more careful.

The points that have tangent lines leading to these points that, if the true value of x could be input into the algorithm, the iteration would crash. Of course, there are points leading to these points, and points leading to those points. There are, in fact, an infinitude of points at which the iteration should crash at some point in the process. Since all are irrational, we generally escape unscathed.

To find these points, we need to solve for x_n in the equation $0.357403 = x_n - \frac{e^x - 4x_n}{e^x - 4}$.



More generally, if we let $x_{n+1} = C$ be our target, we need to solve $C = x_n - \frac{f(x_n)}{f'(x_n)}$,

so $f'(x_n)C = f'(x_n)x_n - f(x_n)$ and $f'(x_n)(C - x_n) + f(x_n) = 0$. We can solve this with Newton's method! If we let $H(x) = f'(x_n)(C - x_n) + f(x_n)$, then

$$x_{n+1} = x_n - \frac{H(x_n)}{H'(x_n)} = x_n - \frac{f'(x_n)(C - x_n) + f(x_n)}{f''(x_n)(C - x_n)}.$$ This iteration will find those points leading to the problematic critical values.

$x_0 := 1.75$
1.75
2.308201
2.121503
2.068418
2.06452
2.0645
2.0645
2.0645
2.0645
2.0645
2.0645

c) On some computers, computation of square roots uses the iterative process $x_{n+1} = \frac{1}{2}\left(x_n + \frac{k}{x_n}\right)$. Show that Newton's method applied to $f(x) = x^2 - k$ can be written in this form.

c) We have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. If we use $f(x) = x^2 - k$ and $f'(x) = 2x$, so $x_{n+1} = x_n - \frac{(x_n)^2 - k}{2(x_n)} = \frac{2x_n^2 - x_n^2 + k}{2x_n} = \frac{1}{2}\left(x_n + \frac{k}{x_n}\right)$. If we let $k = 5$, for example, and begin with $x_0 = 1$, we have $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$ which generates the sequence 1, 2.303571, 2.237057, 2.36068, which is $\sqrt{5}$ to 5 decimal places. Once the sequence converges, so $x_{n+1} = x_n$, we can see that the solution to $x_n = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$ is indeed $x_n = \sqrt{5}$ by solving for x_n.

$\sqrt{5} = 2.236068$

$x_0 := 1$

1
2.303571
2.237057
2.236068
2.236068
2.236068
2.236068
2.236068
2.236068
2.236068
2.236068

d) Use Newton's method to find the zero of $f(x) = \frac{1}{x} - k$. Use your result to explain how you can find reciprocals without ever dividing.

d) As in c), we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. If we use $f(x) = \frac{1}{x} - k$ and $f'(x) = -\frac{1}{x^2}$, then $x_{n+1} = x_n - \frac{\left(\frac{1}{x} - k\right)}{\left(\frac{1}{x^2}\right)} = 2x_n - kx_n^2$. This iteration converges only for $-1 \le k \le 1$. We see that when $x_{n+1} = x_n$ we have $x_n = 2x_n - kx_n^2$, so $x_n = \frac{1}{k}$.

$k := \frac{1}{5}$

$x_0 := 1$

1
1.8
2.952
4.161
4.859
4.996
5
5
5
5
5

e) A student decided to try the iterative equation $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$ to generate a sequence of values $x_0, x_1, x_2, \ldots$. Use several different functions to determine what feature of f occurs at the point this iteration finds.

This is the iteration we used in part b). It finds the zeros of the first derivative, so it searches for critical points.

Basins of Attraction

f) Color the *x*-axis below so that all points in root **A**'s basin of attraction are red, all points in root **B**'s basin of attraction are blue, and all points in root **C**'s basin of attraction are green. Use black for those points that are in none of the basins of attraction. Can you find the point between 2 and 2.1 which separates the basins of attraction for root **B** and root **A**?

f) You can see a bit of the problem with these two tables. If $x_0 = a$ converges to one root and $x_0 = b$ converges to a different root, then there will be some value $x_0 \in (a, b)$ which converges to the third root. This is an example of an infinite complexity. Of course, your computer will give out and numerical errors will creep in and create problems when the intervals get too small.

$x_0 := 2$, $y_0 := 2.05$, $z_0 := 2.1$

$x_i =$	$y_i =$	$z_i =$
2	2.05	2.1
1	0.575572	-0.696176
1.560405	2.703976	-0.552769
1.486814	2.627325	-0.539938
1.487962	2.617998	-0.539835
1.487962	2.617867	-0.539835
1.487962	2.617867	-0.539835
1.487962	2.617867	-0.539835
1.487962	2.617867	-0.539835
1.487962	2.617867	-0.539835
1.487962	2.617867	-0.539835

$x_0 := 2.035$, $y_0 := 2.04$, $z_0 := 2.05$

$x_i =$	$y_i =$	$z_i =$
2.035	2.04	2.05
0.742943	0.692315	0.575572
1.89073	2.041003	2.703976
1.330905	0.681589	2.627325
1.488037	2.080008	2.617998
1.487962	0.025074	2.617867
1.487962	-1.081983	2.617867
1.487962	-0.652904	2.617867
1.487962	-0.546895	2.617867
1.487962	-0.539866	2.617867
1.487962	-0.539835	2.617867

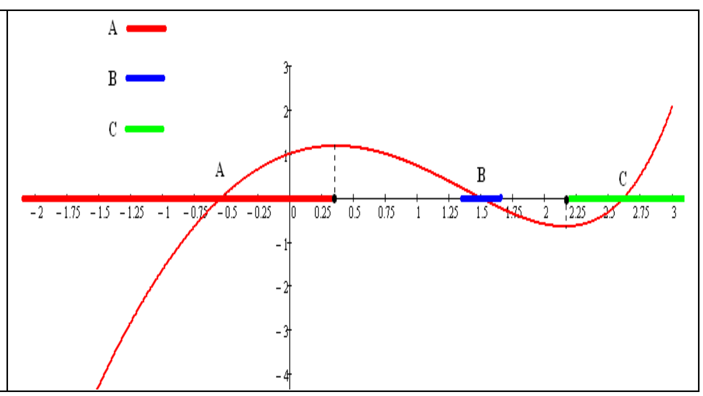
$x_0 := 2.035$, $y_0 := 2.03998$, $z_0 := 2.04$

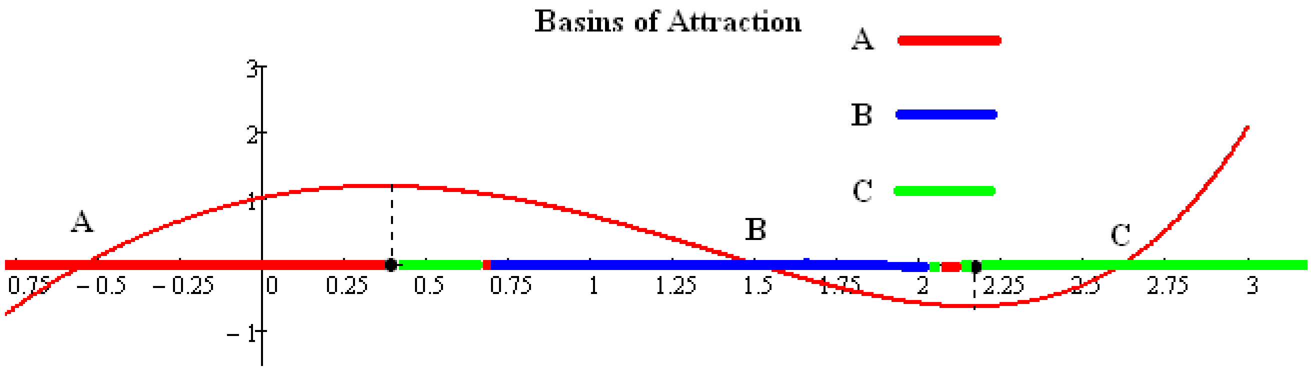
$x_i =$	$y_k =$	$z_i =$
2.035	2.617867	2.04
0.742943	2.617867	0.692315
1.89073	2.617867	2.041003
1.330905	2.617867	0.681589
1.488037	2.617867	2.080008
1.487962	2.617867	0.025074
1.487962	2.617867	-1.081983
1.487962	2.617867	-0.652904
1.487962	2.617867	-0.546895
1.487962	2.617867	-0.539866
1.487962	2.617867	-0.539835

So, you really can't find a definitive answer to where one interval begins and another ends.

But there are some things that should be clear. Any initial value, x_0, to the left of the turning point between root A and root B will be in A's basin of attraction. Similarly, initial values to the right of the turning point between root B and root C. Also, points in a neighborhood of root B will converge to B. The interesting stuff happens in between these locations.

A rough approximation is shown below.





The values less than 0.357 go to A (RED). There is a region around 2.1 that has a tangent line that lands in this region, so it is also RED. There is a region around 0.75 that has a tangent that fall into the RED region near 2, so it must be RED. There must be a region closer to 2 that has a tangent in the RED region by 0.75. I can't find it, but it should be there…ad infinitum.

g) In which intervals is most of the interesting stuff happening? Can you explain why those intervals exist? Zoom in on one of those intervals and color code the axis.

Let's see what happens between 2 and 2.3. At some transition locations, the value of $x_n - \frac{f'(x_n)}{f''(x_n)}$ is too large for the technology. In these cases, the point should be in the basin of attraction for C (if it is too large and positive) or A (if it is too large in absolute value and negative). If we could drill in more closely, we should find a small blue region around 2.065 and a 2.154 and a small green region around 2.04. Welcomed to Chaos.