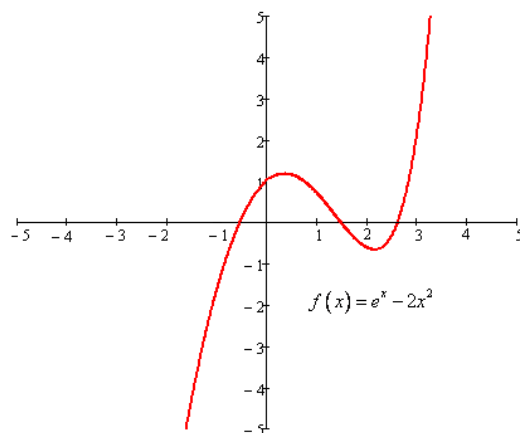


NCAAPMT Calculus Challenge 2010-2011

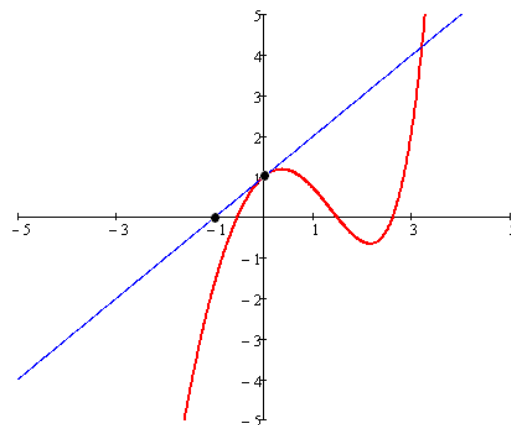
Calculus Challenge #3

Due: Wednesday, October 27

At this point in calculus, you have learned how to write the equation of a line tangent to a curve at a specific point $x = x_0$. The tangent line, $y = f'(x_0)(x - x_0) + f(x_0)$, has many important applications. In this challenge, we will use the tangent line to find approximate solutions to “unsolvable” equations. For example, suppose we want to know the values of x for which $e^x = 2x^2$. The solutions to this equation are the zeros or roots of the function $f(x) = e^x - 2x^2$. The graph of f indicates a solution between $x = -1$ and $x = 1$.



We can approximate the function f with its tangent line at any point we choose. If we pick $x_0 = 0$, the equation of the tangent line is $y = x + 1$. If the tangent line is a good approximation for the function near $x = 0$, then the zero of the tangent line should be a good approximation for the zero of the function near $x = 0$. The root or x -intercept of the tangent line $y = x + 1$ is $x = -1$, so the zero of the function f should be near $x = -1$. In fact, we expect that -1 will be closer to the actual root than was our initial value of $x_0 = 0$.

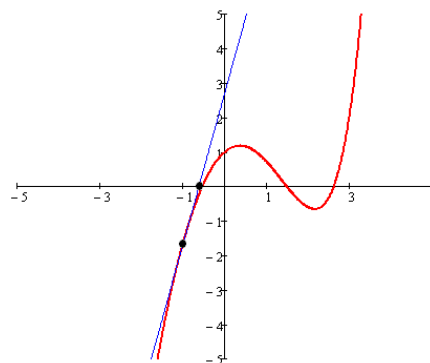


If we now use $x = -1$ as the point of tangency, we should get a better approximation for the root, as you see in the graph at right. If we repeat the process several more times, we can achieve a very good and efficient approximation for the root of the function and thus, for the solution to the equation.

We now generalize the procedure, which is known as Newton's method. Let f be the function for which we are finding a root and let x_0 be the initial x -value, which we believe lies close to a root.

The initial tangent line has the equation

$$y = f'(x_0)(x - x_0) + f(x_0).$$



The x -intercept of this line is found where $y = 0$, so we solve $f'(x_0)(x - x_0) + f(x_0) = 0$ for x . Then $x = x_0 - \frac{f(x_0)}{f'(x_0)}$. If we call this x -intercept x_1 , then we can repeat the process to find $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

The sequence of values, x_0, x_1, x_2, \dots generated by the iterative equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

should converge rapidly to the root we seek. A spreadsheet or calculator can help automate this iterative process.

If a function has more than one root, the root you find depends upon initial value x_0 you choose. Notice in our example, $x_0 = 0$ led to the solution below 0 instead of the one above 0.

First, here are some short problems to get comfortable with Newton's method:

a) Use Newton's method to find the three solutions to the equation $e^x = 2x^2$ to three decimal place accuracy.

b) There are limitations to Newton's method. What initial values of x_0 would keep the method from finding a zero for $f(x) = e^x - 2x^2$? What initial value of x_0 would lead to those values as x_1 ?

c) On some computers, computation of square roots uses the iterative process $x_{n+1} = \frac{1}{2} \left(x_n + \frac{k}{x_n} \right)$.

Show that Newton's method applied to $f(x) = x^2 - k$ can be written in this form.

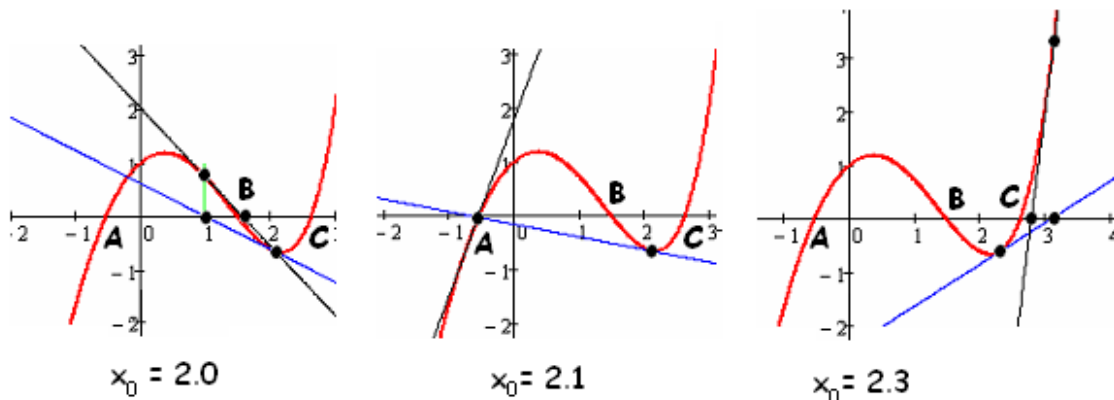
d) Use Newton's method to find the zero of $f(x) = \frac{1}{x} - k$. Use your result to explain how you can find reciprocals without ever dividing.

e) A student decided to try the iterative equation $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$ to generate a sequence of values x_0, x_1, x_2, \dots . Use several different functions to determine what feature of f occurs at the point this iteration finds.

Basins of Attraction

The most interesting aspect of Newton's method stems from determining which starting values are attracted to which root. Our function f has three roots. We will label them **A**, **B**, and **C**. You have approximated each of these in part a). Associated with each of these roots is a set of initial x_0 -values that lead to that root. The intervals on the x -axis that lead to root **A** is called the basin of

attraction for root **A**. For example, if we start at $x_0 = 2$, Newton's method will find root **B**, so 2 is in the basin of attraction for root **B**. If we start at $x_0 = 2.1$, Newton's method will find root **A**. So, 2.1 is in the basin of attraction of root **A**. And if we start at $x_0 = 2.3$, Newton's method will find root **C**, so 2.3 is in the basin of attraction of root **C**.



In the graphs above, the first tangent line is blue and the second is black. As you can see, Newton's method gets very interesting around $x = 2$.

For the function $f(x) = e^x - 2x^2$, the entire x -axis can be split into 4 groups. Those that are in the basin of attraction for root **A**, those that are in the basin of attraction for root **B**, those that are in the basin of attraction for root **C**, and finally, those isolated points that cause Newton's method to fail.

f) Color the x -axis below so that all points in root **A**'s basin of attraction are red, all points in root **B**'s basin of attraction are blue, and all points in root **C**'s basin of attraction are green. Use black for those points that are in none of the basins of attraction. Can you find the point between 2 and 2.1 which separates the basins of attraction for root **B** and root **A**?

g) In which intervals is most of the interesting stuff happening? Can you explain why those intervals exist? Zoom in on one of those intervals and color code the axis.

