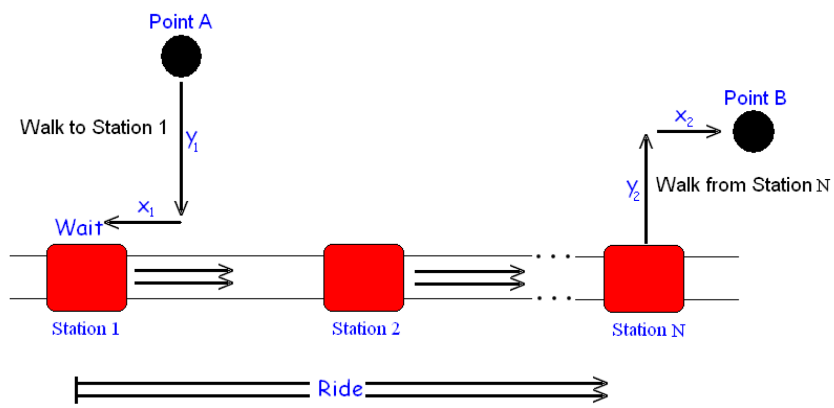
Calculus Challenge #4 A Possible Solution
The Subway Design Problem

Developing the model:

Assumptions must be made to make the problem solvable. We have to decide what aspects of the real-world situation we want to keep and which we can safely ignore. There are lots of ways to do this, so your model need not match mine. This is **a** solution, not **the** solution.



Some assumptions:

- The subway stations are all separated by the same distance.
- The model is intended to minimize the commuting time for a "typical" commuter. It is *not* a probabilistic model that accounts for the distributions of commuters' distances traveled, walking speeds, *etc.*
- The commuter's origin and destination are along the subway route.
- The commuter does not change trains.
- The typical commuter will always walk to the nearest subway station and get off at the station nearest her destination.

We need to consider the times it takes to:

(Walk y1) + (Walk x1) + (Wait for Train) + (Ride a distance R) + (Walk y2) + (Walk x2)

The time for the Ride is split into riding at normal speed and stopping at stations that we pass through. This is where the distance between stations, and consequently, the number of stops, plays a role.

Let's begin by writing down some of the parameters and variables in the model.

T = total duration of trip (the dependent variable)
R = total riding distance separating commuter's home from her destination
D = distance separating subway stops (the independent variable)
w = speed at which person walks
S = speed of train when not accelerating away from or decelerating towards a station
W = time that someone waits for the subway to arrive
k = time lost due per station for decelerating towards it, letting passengers on and off, and accelerating away from it

Do the easy stuff first.

Walk y1 + Walk x1 + (Wait for Train) + (Ride a distance R) + Walk x2 + walk y2

$$T = \frac{y_1}{w} + \frac{x_1}{w} + W + (Ride) + \frac{x_2}{w} + \frac{y_2}{w}.$$

Notice we will think about riding time separately since it is a bit more complicated.

Riding time is composed of the time riding at normal speed , which is $\frac{R}{S}$, plus the extra time to accelerate/decelerate and be stopped at each station, $k \cdot number\ of\ stops$. The number of stops is estimated by $\frac{R}{D}$. This should literally be $\left\lfloor \frac{R}{D} \right\rfloor$, but that would mean our model is not differentiable, so we will gladly give up additional fidelity to reality for differentiability. It turns out, we are often willing to give up quite a lot in our models for differentiability and integrability.

So,

$$T = \frac{y_1}{w} + \frac{x_1}{w} + W + \left(\frac{R}{S} + k \cdot \frac{R}{D} \right) + \frac{x_2}{w} + \frac{y_2}{w}.$$

The independent variable is *D*, so, do we just differentiate this equation with respect to *D*?

Not yet, because some of the other terms in the model also depend on *D*.

The vertical distances (*y*'s) do not depend on *D*. The expected distance we walk horizontally (remember, we are in the city and have to go on the sidewalks) depends on how far apart the stations are. The horizontal distance that one must walk will uniformly distributed between 0 and $\frac{D}{2}$. On average, we will walk a distance of $\frac{D}{4}$. So,

$$T = \frac{y_1}{w} + \frac{D}{4w} + W + \left(\frac{R}{S} + \frac{kR}{D} \right) + \frac{D}{4w} + \frac{y_2}{w}.$$

Note that the max/min ying-yang is a component of this model. *D* is in the numerator of walking time, so decreasing *D* reduces the travel time and *D* is in the denominator of the riding time (related to stops and accel/decel), so increasing *D* reduces the travel time. It is the tug-of-war between these two components that will determine the outcome. All the rest of the model serves merely as interested by-standers.

If $T = \frac{y_1}{w} + \frac{D}{4w} + W + \left(\frac{R}{S} + \frac{kR}{D} \right) + \frac{D}{4w} + \frac{y_2}{w}$, then $\frac{dT}{dD} = \frac{1}{4w} - \left(\frac{kR}{D^2} \right) + \frac{1}{4w}$.

If $\frac{dT}{dD} = 0$, then $\frac{kR}{dD^2} = \frac{1}{2w}$ and

$$D = \sqrt{2wR(k)}.$$

Since $\frac{d^2T}{dD^2} > 0$ always, we know we have the minimum value. Note also that *D* = 0 is a critical value, but does not correspond to a real-world solution.

Does this solution make sense. The optimal distance between stops will increase if the distance to be traveled, *D*, increases, if the walking speed increases, and if the time lost accelerating/decelerating and letting passengers on and off at each stop increases. All those make sense. Notice that neither the waiting time for the train nor the actual riding time at top speed appears in the solution, since they are the same regardless of the distance between stations. Surprisingly, the speed of the train is not a factor.

Based on our model, we would suggest that the distance $D = \sqrt{2wR(k)}$. Note that the units are

$$D = \sqrt{2\left(\frac{\text{Blocks}}{\text{minute}}\right)(\text{Blocks})(\text{minute})} = \text{Blocks}$$

If we use blocks as our basic unit of distance, then $w \approx 1.33\, {}^{Bl}\!/_{\min}$, $W \approx 7\min$, $R \approx 167Bl$, $S \approx 11.2\, {}^{Bl}\!/_{\min}$, and $k \approx 0.9\min$ per stop.

So, $D = \sqrt{2(1.33)(167)(0.9)} \approx 20$ block should be optimal for a 167 block trip with the parameter values for speeds given.

The "controllable" part of the trip, $T = \frac{D}{2w} + \left(\frac{R}{S} + \frac{kR}{D}\right)$, which is related to the distance between stops, should take about

$$T = \frac{20}{2(1.33)} + \left(\frac{167}{11.3} + \frac{(0.9)167}{20}\right) \approx 7.5 + (14.8 + 7.5) = 29.8 \text{ minutes.}$$

If the stations are places every 10 blocks, and everything else is the same, the trip will take $T = \frac{10}{2(1.33)} + \left(\frac{167}{11.3} + \frac{(0.9)167}{10}\right) \approx 3.75 + (14.8 + 15) = 33.55$ minutes. You can see that we saved time walking but lost time with the stops.