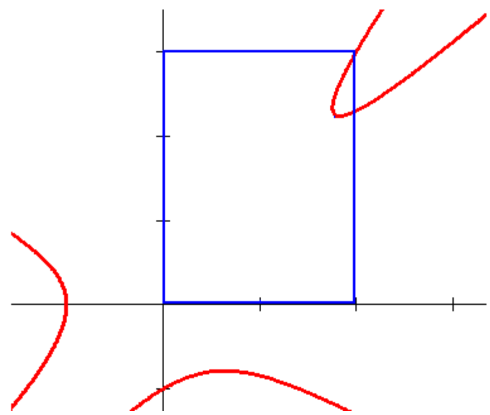
Calculus Challenge Problems #5

Related Rates

This week's challenge is more traditional than the modeling problem last week. It is a collection of related rates problems that don't quite fit anywhere else, so I thought I would put them together in a single challenge. Note that the solutions are due after Thanksgiving week, so you have three weeks to do this challenge.

1. A rectangle has two sides along the positive coordinate axes, and its upper right corner lies on the curve $x^3 - 2xy^2 + y^3 + 1 = 0$. How fast is the area of this rectangle changing as the point passes the position (2, 3) if it is moving so that $\frac{dx}{dt} = 1$ unit per second?



We want to determine the rate of change of $A = xy$ at (2,3) along the path $x^3 - 2xy^2 + y^3 + 1 = 0$.

So, we need $\frac{dA}{dt} = \left(\frac{dx}{dt}\right)y + x\left(\frac{dy}{dt}\right)$ with $x = 2$, $y = 3$, and $\frac{dx}{dt} = 1$. All that is missing is $\frac{dy}{dt}$.

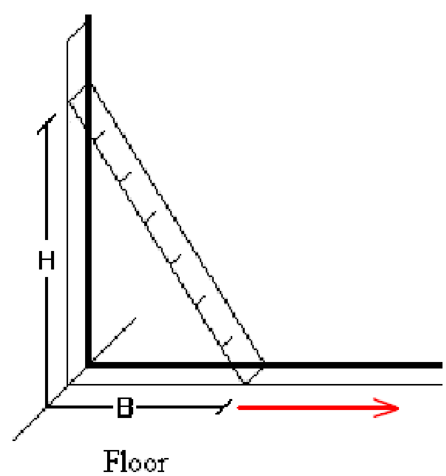
We have the path $x^3 - 2xy^2 + y^3 + 1 = 0$, so

$$3x^2\left(\frac{dx}{dt}\right) - 2\left(\frac{dx}{dt}y^2 + 2xy\frac{dy}{dt}\right) + 3y^2\left(\frac{dy}{dt}\right) = 0, \text{ and } \frac{dy}{dt} = \frac{\left(3x^2 - 2y^2\right)\left(\frac{dx}{dt}\right)}{4xy - 3y^2} = \frac{(3\cdot 4 - 2\cdot 9)\cdot 1}{4\cdot 6 - 3\cdot 9} = 7.$$

The area of the rectangle is increasing at a rate of 7 units^2 /sec.

2. By now I'm sure you have done the problem of a ladder sliding down the wall. Just in case you haven't done this problem yet, I'll give you the opportunity now.

If a meter stick slides down a wall so that the bottom of the meter stick moves away from the wall at a constant speed of 5 cm/sec, according to your model, how far above the ground is the top of the meter stick when it breaks the sound barrier?



Here we have $x^2 + y^2 = 100^2$, so $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ and $\frac{dy}{dt} = -\left(\frac{x}{y}\right)\frac{dx}{dt}$. We know that $\frac{dx}{dt} = 5$ cm/sec and that the sound barrier is at 34320 cm/sec (at one atmosphere dry air and 20 degrees Celsius), so we want to solve for *y* in $34320 = 5\left(\frac{\sqrt{100^2 - y^2}}{y}\right)$. Your value for the speed of sound may vary, depending upon your reference.

So, $6864y = \sqrt{100^2 - y^2}$ and $\left(6864^2 + 1\right)y^2 = 100^2$, so $y = 0.01457$ cm from the bottom.

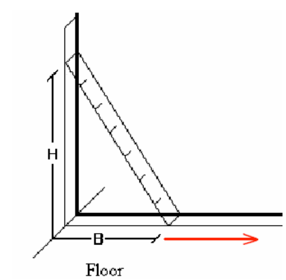
3. **The Extension Ladder Extension**

One nice extension to the classic ladder problem is to consider an extension ladder of length *L* that can contract and elongate (it is an extension ladder so it can do this) as it slides down the wall. The ladder starts out flat along the wall and the foot of the ladder is moving away from the ladder at a constant speed, *s*. Now suppose the ladder moves down the wall in such a way that *the distance the top moves down the wall is equal to the distance the bottom has moved away from the wall.* In order for this to happen, the ladder needs to extend and contract as it slides down the wall. As the ladder slides to the ground,

a) When is the ladder contracting and when is it extending?

b) Where is the ladder the longest and where is it the shortest?

Now, we have $x^2 + y^2 = L^2$ where *x*, *y*, and *L* are all functions of time. Also, we have $\frac{dx}{dt} = -\frac{dy}{dt} = s$ We are interested in $\frac{dL}{dt}$.



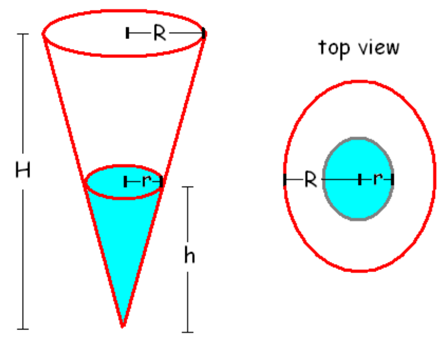
So, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2L\frac{dL}{dt}$ and $\frac{dL}{dt} = \frac{(x-y)s}{L} = \frac{(x-y)s}{\sqrt{x^2+y^2}}$.

$\frac{dL}{dt}$ is positive whenever $x > y$ and negative whenever $x < y$. So, the ladder is decreasing in length from the beginning until it is at a 45 degree angle, then the length of the ladder increases until it hits the ground. Since it is of length L_0 at the beginning and end, these must be the maximum length.

Since the derivative goes from negative to positive when $x = y$ there is a minimum length here. We can also see this using the second derivative test.

$\frac{dL}{dt} = \frac{(x-y)s}{L}$, so $\frac{d^2L}{dt^2} = \frac{L\left(\left(x-y\right)\frac{ds}{dt} + \left(\frac{dx}{dt} - \frac{dy}{dt}\right)s\right) - (x-y)s\left(\frac{dL}{dt}\right)}{L^2}$. When $x = y$, then $\frac{d^2L}{dt^2} = \frac{L\left(0 + (s+s)s\right) - 0}{L^2} = \frac{2s^2}{L} > 0$.

4. A thunderstorm is dropping rain at the rate of k inches per hour into a conical tank of water of radius R feet and height H feet. Let $\frac{dh}{dt}$ represent the rate at which the depth of the water in the cone is increasing. Show that the ratio of $\frac{dh}{dt}$ to the rate of rainfall is equal to the ratio between the area of the tank's opening to the area of the water surface.



We know that $V = \frac{\pi}{3} r^2 h$ and $\frac{r}{R} = \frac{h}{H}$, so $r = \frac{Rh}{H}$. Then $V = \frac{\pi}{3}\left(\frac{R}{H}\right)^2 h^3$ and $\frac{dV}{dt} = \pi\left(\frac{R}{H}\right)^2 h^2 \left(\frac{dh}{dt}\right)$.

The rain is falling at a rate of k inches per hour, so the volume of water entering the top of the cone is $k \cdot$(area of the top). So, another form of $\frac{dV}{dt}$ is $\frac{dV}{dt} = k\left(\pi R^2\right)$.

Then $k\left(\pi R^2\right) = \pi\left(\frac{R}{H}\right)^2 h^2 \left(\frac{dh}{dt}\right)$. Rearranging, we have $\frac{k}{\left(\frac{dh}{dt}\right)} = \frac{\left(\pi\left(\frac{R}{H}\right)^2 h^2\right)}{\left(\pi R^2\right)} = \frac{h^2}{H^2}$.

But we know that $\frac{r}{R} = \frac{h}{H}$, so $\frac{k}{\left(\frac{dh}{dt}\right)} = \frac{r^2}{R^2} = \frac{\pi r^2}{\pi R^2}$ as desired.