

Calculus Challenge Problems #7

Solutions due February 2

Given two distinct positive real numbers, a and b , we know how to compute their average. In fact, you may have run into several different kinds of averages or means that are useful in special situations. Here is a partial list of means that have applications in a variety of areas in engineering and science.

The arithmetic mean $A(a,b) = \frac{a+b}{2}$.

The geometric mean, $G(a,b) = \sqrt{a \cdot b}$

The harmonic mean, $H(a,b) = \frac{2ab}{a+b}$.

The logarithmic mean, $L(a,b) = \frac{b-a}{\ln(b) - \ln(a)}$.

The Heronian mean, $N(a,b) = \frac{a + \sqrt{ab} + b}{3}$.

The centroidal mean, $C(a,b) = \frac{2(a^2 + ab + b^2)}{3(a+b)}$.

Reference: Chen, Hongwei, *Means Generated by Integral*, **Mathematics Magazine**, Vol. 78, No. 5, December, 2005, page 397-399.

a) Which means can be equal and under what conditions?

Except for the logarithmic mean, they are all equivalent when $a = b$.

$$A(a,a) = \frac{2a}{2} = a, \quad G(a,a) = \sqrt{a \cdot a} = |a| = a \text{ (since } a > 0), \quad H(a,a) = \frac{2a^2}{2a} = a,$$

$$N(a,a) = \frac{a + \sqrt{a^2} + a}{3} = \frac{2a + |a|}{3} = a \text{ (since } a > 0), \text{ and } C(a,a) = \frac{2(a^2 + a^2 + a^2)}{3(2a)} = \frac{3a^2}{3a} = a.$$

The function $f(t) = \frac{\int_a^b x^{t+1} dx}{\int_a^b x^t dx}$ gives each of the formulas above for some rational value of t .

b) Show that when $t = -\frac{1}{2}$, the value of $f(t) = \frac{\int_a^b x^{t+1} dx}{\int_a^b x^t dx}$ is the Heronian mean.

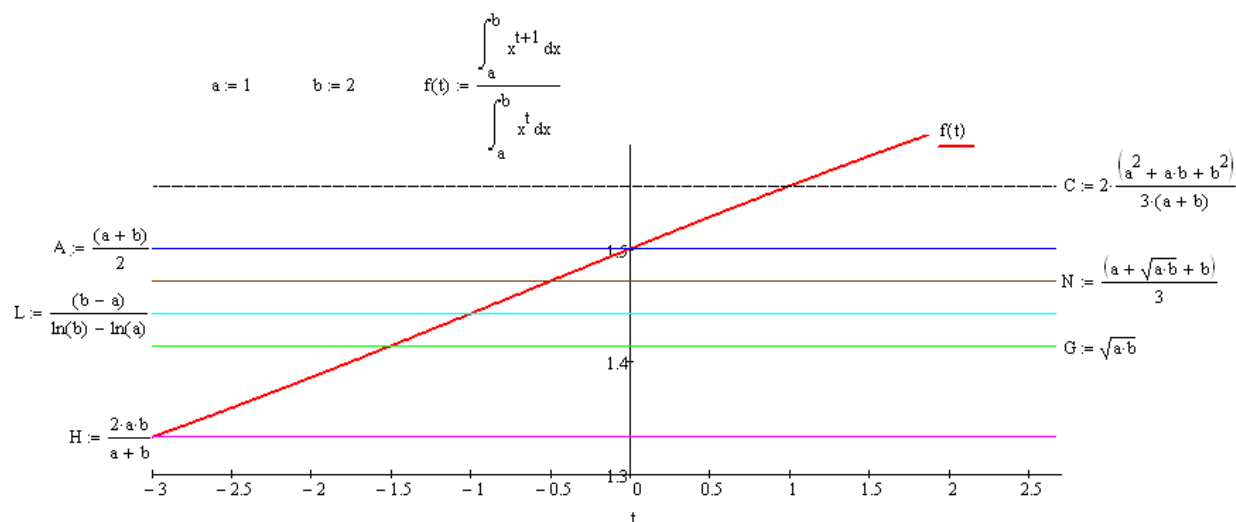
For each of these problems, we need to recall that differences of powers always factor.

If $t = -\frac{1}{2}$, then

$$f(t) = \frac{\int_a^b x^{\frac{1}{2}} dx}{\int_a^b x^{-\frac{1}{2}} dx} = \frac{\left(\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_{x=a}^{x=b}}{\left(2x^{\frac{1}{2}}\right)\Big|_{x=a}^{x=b}} = \left(\frac{1}{3}\right) \frac{b^{\frac{3}{2}} - a^{\frac{3}{2}}}{b^{\frac{1}{2}} - a^{\frac{1}{2}}} = \left(\frac{1}{3}\right) \frac{(b^{\frac{1}{2}} - a^{\frac{1}{2}})(b + (ba)^{\frac{1}{2}} + a)}{(b^{\frac{1}{2}} - a^{\frac{1}{2}})} = \frac{a^2 + \sqrt{ab} + b^2}{3}.$$

c) Find rational values for t , $-3 \leq t \leq 3$, which generate each of the means defined above.

After trying several values, I decided to use a graph to help. By looking at the graph below, I can estimate the values and then show they are correct by integration.



Based on the graph, we conjecture that we have :

The centroidal mean (C) when $t = 1$.

The arithmetic mean (A) when $t = 0$.

The Heronian mean (N) when $t = -\frac{1}{2}$.

The logarithmic mean (L) when $t = -1$.

The geometric mean (G) when $t = -\frac{3}{2}$.

The harmonic mean (H) when $t = -3$.

To verify these conjectures, we do the integration.

The centroidal mean (C) when $t = 1$.

$$f(t) = \frac{\int_a^b x^2 dx}{\int_a^b x^1 dx} = \frac{\left(\frac{1}{3}x^3 \Big|_{x=a}^{x=b}\right)}{\left(\frac{1}{2}x^2 \Big|_{x=a}^{x=b}\right)} = \left(\frac{2}{3}\right) \frac{b^3 - a^3}{b^2 - a^2} = \left(\frac{2}{3}\right) \frac{(b-a)(b^2 + ab + a^2)}{(b-a)(b+a)} = \frac{2(b^2 + ab + a^2)}{3(b+a)}.$$

The arithmetic mean (A) when $t = 0$.

$$f(t) = \frac{\int_a^b x^1 dx}{\int_a^b x^0 dx} = \frac{\left(\frac{1}{2}x^2 \Big|_{x=a}^{x=b}\right)}{\left(x \Big|_{x=a}^{x=b}\right)} = \left(\frac{1}{2}\right) \frac{b^2 - a^2}{b - a} = \left(\frac{1}{2}\right) \frac{(b-a)(b+a)}{(b-a)} = \frac{a+b}{2}.$$

The logarithmic mean (L) when $t = -1$.

$$f(t) = \frac{\int_a^b x^0 dx}{\int_a^b x^{-1} dx} = \frac{\left(x \Big|_{x=a}^{x=b}\right)}{\left(\ln|x| \Big|_{x=a}^{x=b}\right)} = \frac{b-a}{\ln(b) - \ln(a)}.$$

The geometric mean (G) when $t = -\frac{3}{2}$.

$$f(t) = \frac{\int_a^b x^{-\frac{1}{2}} dx}{\int_a^b x^{-\frac{3}{2}} dx} = \frac{\left(2x^{\frac{1}{2}} \Big|_{x=a}^{x=b}\right)}{\left(-2x^{-\frac{1}{2}} \Big|_{x=a}^{x=b}\right)} = \frac{(-1)(\sqrt{b} - \sqrt{a})}{\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}\right)} = \frac{(-1)(\sqrt{b} - \sqrt{a})}{\left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}\right)} = \frac{\sqrt{ab}(\sqrt{b} - \sqrt{a})}{(-1)(\sqrt{a} - \sqrt{b})} = \sqrt{ab}.$$

The harmonic mean (H) when $t = -3$

$$f(t) = \frac{\int_a^b x^{-2} dx}{\int_a^b x^{-3} dx} = \frac{\left(-x^{-1} \Big|_{x=a}^{x=b}\right)}{\left(\frac{-1}{2}x^{-2} \Big|_{x=a}^{x=b}\right)} = (2) \frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)} = (2) \frac{\left(\frac{a-b}{ab}\right)}{\left(\frac{a^2 - b^2}{a^2 b^2}\right)} = (2) \left(\frac{a-b}{ab}\right) \left(\frac{a^2 b^2}{a^2 - b^2}\right) = \frac{2ab}{a+b}.$$

d) What is $\lim_{t \rightarrow \infty} f(t)$ and $\lim_{t \rightarrow -\infty} f(t)$? Explain your reasoning.

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} \frac{\int_a^b x^{t+1} dx}{\int_a^b x^t dx} = \lim_{t \rightarrow \infty} \frac{\left(\frac{b^{t+2} - a^{t+2}}{t+2} \right)}{\left(\frac{b^{t+1} - a^{t+1}}{t+1} \right)} = \lim_{t \rightarrow \infty} \left(\frac{t+1}{t+2} \right) \left(\frac{b^{t+2} - a^{t+2}}{b^{t+1} - a^{t+1}} \right).$$

We know that $\lim_{t \rightarrow \infty} \left(\frac{t+1}{t+2} \right) = 1$, so we only need to consider $\lim_{t \rightarrow \infty} \left(\frac{b^{t+2} - a^{t+2}}{b^{t+1} - a^{t+1}} \right)$. Since we have $a < b$, we

can write $b = ka$ for some $k > 1$. Then

$$\lim_{t \rightarrow \infty} \left(\frac{k^{t+2} a^{t+2} - a^{t+2}}{k^{t+1} a^{t+1} - a^{t+1}} \right) = \lim_{t \rightarrow \infty} \left(\frac{a^{t+2}}{a^{t+1}} \right) \left(\frac{k^{t+2} - 1}{k^{t+1} - 1} \right) = \lim_{t \rightarrow \infty} (a) \left(\frac{k^{t+2} - 1}{k^{t+1} - 1} \right) = a \cdot k. \text{ But } b = ka, \text{ so the limit is } b.$$

Similarly, we have

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} \frac{\int_a^b x^{t+1} dx}{\int_a^b x^t dx} = \lim_{t \rightarrow \infty} \left(\frac{t+1}{t+2} \right) \left(\frac{b^{t+2} - a^{t+2}}{b^{t+1} - a^{t+1}} \right) = \lim_{t \rightarrow \infty} \left(\frac{a^{t+2}}{a^{t+1}} \right) \left(\frac{k^{t+2} - 1}{k^{t+1} - 1} \right) = \lim_{t \rightarrow \infty} (a) \left(\frac{k^{t+2} - 1}{k^{t+1} - 1} \right) = a \cdot 1 = a.$$

The limiting values, as one might expect, are a and b . This proves that all values of the mean are less than the larger value and greater than the smaller value.

For parts e) and f), use $a = 1$ and $b = 2$.

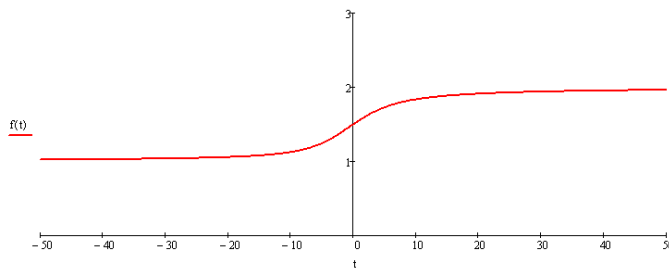
e) Use technology to help you sketch the graph of function f . Note that f is strictly increasing. Use this fact (we won't prove it, since it is really messy and difficult) to order the different means.

From the graph above, we see that $C \geq A \geq N \geq L \geq G \geq H$.

f) Let $f(t) = \frac{\int_a^b x^{t+1} dx}{\int_a^b x^t dx}$. What is the mean of all possible means of $a = 1$ and $b = 2$? Approximate

the average value of $f(t)$ on $(-\infty, \infty)$. Give a rationale for your approximation.

I don't expect that you will be able to get this one rigorously, but I am hoping you will have a couple of good ideas on how to approach it. That's why I asked only for an approximation and a rationale for that approximation.



First, let's make a conjecture. Based on the graph at right, it appears that the function is "symmetricish" about the point (0, 1.5).

This suggests that if an average value exists, it should be the value at $t = 0$, which is the old favorite, the arithmetic mean.

To find the average value, we need to evaluate the multiply improper ratio $Avg\ Value\ of\ f = \frac{\int_{-\infty}^{\infty} f(t) dt}{\int_{-\infty}^{\infty} dt}$

if possible. Notice, this uses the arithmetic mean in the definition of average value.

We can approximate this value numerically:

I've found this to be a really interesting problem to think about. Here is my analytic solution.

There may be some technical details that I've missed, but I think what follows is correct.

$$A(K) := \frac{\int_{-K}^K \frac{\int_1^2 x^{t+1} dx}{\int_1^2 x^t dx} dt}{\int_{-K}^K 1 dt}$$

$A(10) = 1.488$
 $A(100) = 1.49$
 $A(1000) = 1.498$

We need the values of $\lim_{k \rightarrow -\infty} \frac{\int_k^0 f(t) dt}{\int_k^0 dt}$ and $\lim_{n \rightarrow \infty} \frac{\int_0^n f(t) dt}{\int_0^n dt}$. None of these improper integrals converge, so

the limit of the numerators are both infinity and the limits of the denominators are similarly infinity.

This gives us the indeterminate form $\frac{\infty}{\infty}$, so L'Hopital's rule may be a way out.

By L'Hopital's rule, we have $\lim_{n \rightarrow \infty} \frac{\int_0^n f(t) dt}{\int_0^n dt} = \lim_{n \rightarrow \infty} \frac{\frac{d}{dn} \left(\int_0^n f(t) dt \right)}{\frac{d}{dn} \left(\int_0^n dt \right)}$ if this limit exists. By the 2nd

Fundamental Theorem of Calculus, we have $\lim_{n \rightarrow \infty} \frac{f(n)}{1}$. We have shown in d) that this limit is 2. By

similar argument, $\lim_{k \rightarrow -\infty} \frac{\int_0^k f(t) dt}{k} = 1$. So, the question is, how do we combine these two values.

The mean value for positive t is 2 and the mean value for negative t is 1. How do we put them together? I don't actually know if it is legitimate to do this, but since each component should have the same weight, we could simply average the means. But with which average? If we use the arithmetic mean, then the average value would be 1.5. The arithmetic mean would be the mean of all possible means which is suggested our numerical work and our graph.

Of course, if we choose to use the geometric mean to combine the two components, then the geometric mean would be the mean of all possible means. Whichever mean we choose will be the mean of all possible means.