Calculus Challenge Problems #7

Solutions due February 2

Given two distinct positive real numbers, *a* and *b*, we know how to compute their average. In fact, you may have run into several different kinds of averages or means that are useful in special situations. Here is a partial list of means that have applications in a variety of areas in engineering and science.

- The arithmetic mean $A(a,b)=\dfrac{a+b}{2}$.
- The geometric mean, $G(a,b)=\sqrt{a\cdot b}$
- The harmonic mean, $H(a,b)=\dfrac{2ab}{a+b}$
- The logarithmic mean, $L(a,b)=\dfrac{b-a}{\ln(b)-\ln(a)}$
- The Heronian mean, $N(a,b)=\dfrac{a+\sqrt{ab}+b}{3}$.
- The centroidal mean, $C(a,b)=\dfrac{2\left(a^2+ab+b^2\right)}{3(a+b)}$.

a) Which means can be equal and under what conditions?

One natural question to ask about these different means concerns their relative size. You may have learned that $A(a,b)\geq G(a,b)$, the arithmetic mean is always greater than or equal to the geometric mean. How do the other means fit in this inequality? For any two positive real numbers *a* and *b*, with $a < b$, what inequality relates $A(a,b)$, $G(a,b)$, $H(a,b)$, $L(a,b)$, $N(a,b)$, and $C(a,b)$? To answer this question, we can employ the function *f* below.

The function $f(t)=\dfrac{\int_a^b x^{t+1}dx}{\int_a^b x^t dx}$ gives each of the formulas above for some rational value of *t*.

b) Show that when $t = -\frac{1}{2}$, the value of $f(t) = \frac{\int_a^b x^{t+1}dx}{\int_a^b x^t dx}$ is the Heronian mean.

c) Find rational values for *t*, $-3 \le t \le 3$, which generate each of the means defined above.

d) What is $\lim_{t \to \infty} f(t)$ and $\lim_{t \to -\infty} f(t)$? Explain your reasoning.

For parts e) and f), use $a = 1$ and $b = 2$.

e) Use technology to help you sketch the graph of function *f*. Note that *f* is strictly increasing. Use this fact (we won't prove it, since it is really messy and difficult) to order the different means.

f) What is the mean of all possible means of $a = 1$ and $b = 2$? Approximate the average value of $f(t)$ on $(-\infty, \infty)$. Give a rationale for your approximation.