Calculus Challenge Problems #8

Solutions due February 16

Now that you are learning how to integrate, we can revisit the definition of derivative. This bi-week we will consider a variation on the definition of derivative. This derivative is useful in areas of applied statistics and engineering. When I present the solution in two weeks, I'll show you how the equation is derived from a least squares estimate of the slope. For now, we will consider whether the derivative gives us familiar results to our classic derivative.

The ***L***-derivative is an alternate way of defining a derivative of a function at a point $x = a$. The ***L***-derivative, denoted here by $f_L'(a)$, is defined by

$$f_L'(a) = \lim_{h \to 0^+} \frac{3}{2h^3} \int_{-h}^{h} t\, f(a+t)\, dt$$

provided the limit exists.

a) Use this definition to find the ***L***-derivative of x, x^2, and x^3 at $x = a$. Generalize to find the ***L***-derivative of x^n where n is a positive integer,

b) Use the definition of the ***L***-derivative to find the derivative of $|x|$ at $x = a$.
How about at $x = 0$?

c) Use this definition and integration by parts to find the ***L***-derivative of the transcendental functions $f(x) = e^x$ and $f(x) = \sin(x)$ and the rational function $f(x) = \frac{1}{x}$. After doing the integration, you will also need to use L'Hopital's rule. When using L'Hopital's rule, we will fudge a bit and use the "standard" derivative formulas. You have to be very careful with your algebra, but if you are, it works out to be pretty cool. (This won't be graded, but, if you are really bold, try $\ln(x)$ or $\sqrt{x}$)

d) Use L'Hopital's rule evaluate the limit $\lim_{h \to 0^+} \frac{3\int_{-h}^{h} t \cdot f(a+t)\, dt}{2h^3}$. Where have you seen this result before? Does this help explain why this strange integration problem actually computes derivatives?