Calculus Challenge Problems #11

Solutions due March 30

From ALL THINGS CONSIDERED on NPR News (National Public Radio)
August 18, 2007 DEBBIE ELLIOTT, host:

ELLIOTT: As we pursued the moth-flame connection for this week's *Science Out of the Box* segment, we found a few theories out there. We turn to Dr. May Berenbaum, head of the department of entomology at the University of Illinois.

ELLIOTT: So why do moths have this suicidal attraction to flames and light bulbs?

Dr. BERENBAUM: Well, the sort of standard line of explanation is that it's something of an evolutionary short circuit that moths and other nocturnal insects use celestial navigation for orienting about in the dark, the same way that explorers could find their way by charting a course relative to the North Star or some celestial far-distant point source of light. And what has happened since that time is that humans have come along and developed terrestrial point sources of light. So very intense light that is not millions of miles away.

So in a behavior called transverse orientation, many animals, including insects, can move or *fly to maintain a constant angle relative to a distant point source of light.* (Italics added)

A moth is approaching a point of light located at the origin so that the tangent to its path makes a constant angle of α with the light. The moth begins its descent and travels at constant rate along this path. We want to determine how long it will take the moth to reach the light.

First, you need to know something about polar and parametric curves that you may not have studied in class.

Recall that $r = f(\theta)$ is the equation of a curve in polar coordinates. If we want to know the equation of a line tangent to a polar curve at some point, we need to rewrite the polar equation in some Cartesian form. The easiest way to do this is to use the parametric equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

a) Let f be a differentiable function of θ for the polar curve $r = f(\theta)$. Show that

$$\frac{dy}{dx} = \frac{r + \tan(\theta)\frac{dr}{d\theta}}{-r\tan(\theta) + \frac{dr}{d\theta}}.$$

This is the slope of the line tangent to the curve at (x, y) where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

We know that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$. Since $r = f(\theta)$, we have $\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = \frac{dr}{d\theta}\cos\theta - r\sin\theta$ and $\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{dr}{d\theta}\sin\theta + r\cos\theta$. So, $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$. Dividing numerator and denominator by $\cos(\theta)$ gives the result we want: $\frac{dy}{dx} = \frac{r + \tan(\theta)\frac{dr}{d\theta}}{-r\tan(\theta) + \frac{dr}{d\theta}}$.

b) If $f'(\theta)$ is continuous, show that the arc length of the polar curve $r = f(\theta)$ from θ_1 to θ_2 is $L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

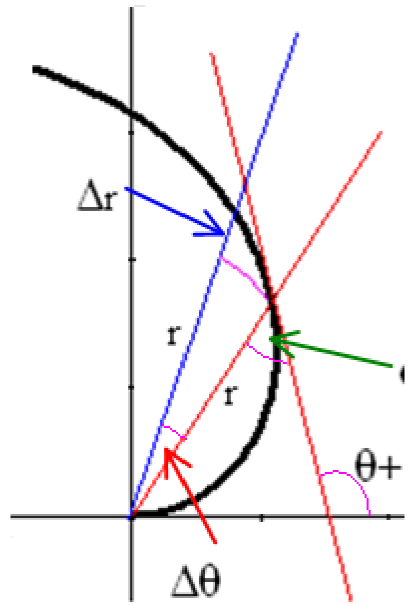
We know that the arc length in parametric form is $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. With theta as the independent variable, this becomes $L = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$. Using the results from a), we have $L = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta.$

We can simplify the expression in the radical by simplifying the Pythagorean identity as noticing that the middle terms add out.

$$\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2 .$$

So, $L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Now that those two equations are known, we go back to the moth. Consider the diagram at right.



c) Since $\frac{dy}{dx} = \tan(\theta + \alpha)$, show that $\frac{dr}{d\theta} = \frac{r}{\tan(\alpha)}$

From the diagram at right, we see that $\frac{dy}{dx} = \tan(\theta + \alpha)$. So,

$$\frac{dy}{dx} = \frac{\tan(\theta) + \tan(\alpha)}{1 - \tan(\theta)\cdot\tan(\alpha)}.$$

From part a), we also know that $\frac{dy}{dx} = \frac{r + \tan(\theta)\frac{dr}{d\theta}}{-r\tan(\theta) + \frac{dr}{d\theta}}$,

so $\frac{\tan(\theta) + \tan(\alpha)}{1 - \tan(\theta)\cdot\tan(\alpha)} = \frac{r + \tan(\theta)\frac{dr}{d\theta}}{-r\tan(\theta) + \frac{dr}{d\theta}}$. We just need to get out our algebra hammer and beat $\frac{dr}{d\theta} = \frac{r}{\tan(\alpha)}$ out of this equation.

If $\frac{\tan(\theta) + \tan(\alpha)}{1 - \tan(\theta)\cdot\tan(\alpha)} = \frac{r + \tan(\theta)\frac{dr}{d\theta}}{-r\tan(\theta) + \frac{dr}{d\theta}}$, then

$$\left(\tan(\theta) + \tan(\alpha)\right)\left(-r\tan(\theta) + \frac{dr}{d\theta}\right) = \left(r + \tan(\theta)\frac{dr}{d\theta}\right)\left(1 - \tan(\theta)\cdot\tan(\alpha)\right)$$

Multiplying out both sides gives

$$-r\tan^2\theta - r\tan\theta\tan\alpha + \frac{dr}{d\theta}\tan\theta + \frac{dr}{d\theta}\tan\alpha = r + \frac{dr}{d\theta}\tan\theta - r\tan\theta\tan\alpha - \tan^2\theta\tan\alpha\frac{dr}{d\theta}$$

Common terms add out, so $-r\tan^2\theta+\frac{dr}{d\theta}\tan\alpha=r-\tan^2\theta\tan\alpha\frac{dr}{d\theta}$. Regrouping, we have $\frac{dr}{d\theta}\left(1+\tan^2\theta\right)\tan\alpha=r\left(1+\tan^2\theta\right)$, so $\frac{dr}{d\theta}=\frac{r}{\tan(\alpha)}$.

d) Solve $\frac{dr}{d\theta}=\frac{r}{\tan(\alpha)}$ for *r* as a function of θ.

This is the classical exponential function of the form $\frac{dy}{dt}=ky$ which has the solution $y=Ae^{kt}$ so $r(\theta)=A_0e^{\frac{\theta}{\tan\alpha}}$. We need an initial condition to find the constant A_0. The value of A_0 determines how rapidly the curve spreads out, and thus the distance to the origin.

e) Given $r=f(\theta)$ from d), determine the distance the moth must travel to reach the light at the origin given that $\alpha=30^\circ$ and it is beginning its flight at $\theta=\frac{\pi}{4}$. It is important to note the values of θ when the is the moth approaching the light.

As θ increases, the moth gets farther and farther from the flame. The moth approaches the flame as $\theta\to-\infty$. Graph the function using Winfeed or your calculator to see its very slow approach. Zoom in around the origin.

Since the moth approaches the flame as $\theta\to-\infty$, so we are interested in $\int_{-\infty}^{\frac{\pi}{4}}\sqrt{\left(A_0e^{\sqrt{3}\theta}\right)^2+\frac{\left(A_0e^{\sqrt{3}\theta}\right)^2}{3}}\,d\theta=A_0\int_{-\infty}^{\frac{\pi}{4}}\sqrt{\frac{4\left(e^{\sqrt{3}\theta}\right)^2}{3}}\,d\theta=\frac{2A_0}{\sqrt{3}}\int_{-\infty}^{\frac{\pi}{4}}e^{\sqrt{3}\theta}\,d\theta$. This is an improper integral whose value is $\frac{2A_0}{\sqrt{3}}\int_{-\infty}^{\frac{\pi}{4}}e^{\sqrt{3}\theta}\,d\theta=\lim_{k\to-\infty}\frac{2A_0}{3}e^{\sqrt{3}\theta}\Big|_k^{\frac{\pi}{4}}=\lim_{k\to-\infty}\frac{2A_0}{3}e^{\frac{\pi\sqrt{3}}{4}}-\frac{2A_0}{3}e^{\sqrt{3}k}=\frac{2A_0}{3}e^{\frac{\pi\sqrt{3}}{4}}$ cm.

If you double the value of A_0, then you double the distance needed to be traveled.

f) How long will it take the moth to reach the light if it is traveling at a constant rate of 1 cm/sec?

At a rate of 1 cm/sec, it will take $\frac{2A_0}{3}e^{\frac{\pi\sqrt{3}}{4}}$ seconds for the moth to reach the flame. For example, if $A_0=1$, this will be 2.598 seconds.