

NCAAPMT Calculus Challenge 2010-2011

Challenge #1 Due: September 29, 2010

All answers must be supported.

Consider the recursively defined function $f(x) = \begin{cases} x-10 & \text{if } x > 100 \\ f(f(x+11)) & \text{if } x \leq 100 \end{cases}$.

a) Find the values of $f(92)$, $f(85)$, $f(92.5)$ and $f(30\pi)$.

This is a bit tedious until you see what is happening. Using the definition, we have

$$\begin{aligned} f(92) &= f(f(103)) = f(93) = f(f(104)) = f(94) = f(f(105)) = f(95) = f(f(106)) = f(96) \\ &= f(f(107)) = f(97) = f(f(108)) = f(98) = f(f(109)) = f(99) = f(f(110)) = f(100) \\ &= f(f(111)) = f(101) = 91 \end{aligned}$$

So, $f(92) = 91$ as is also $f(93)$, $f(94)$, $f(95)$, $f(96)$, $f(97)$, $f(98)$, $f(99)$, $f(100)$, and $f(101)$.

Similarly, we have

$$\begin{aligned} f(85) &= f(f(95)) = f(f(f(106))) = f(f(96)) = f(f(f(107))) = f(f(97)) = f(f(f(108))) \\ &= f(f(98)) = f(f(f(109))) = f(f(99)) = f(f(f(110))) = f(f(100)) = f(f(f(111))) \\ &= f(f(101)) = f(f(91)) = f(f(f(102))) = f(f(92)) \end{aligned}$$

Aha! We know that $f(92) = 91$, so $f(f(92)) = f(91)$ and $f(91) = f(f(102)) = f(92) = 91$.

So, $f(85) = 91$.

Now, what about non-integers. We can use the string generated by $f(92)$

$$\begin{aligned}
f(92.5) &= f(f(103.5)) = f(93.5) = f(f(104.5)) = f(94.5) = f(f(105.5)) = f(95.5) \\
&= f(f(106.5)) = f(96.5) = f(f(107.5)) = f(97.5) = f(f(108.5)) = f(98.5) \\
&= f(f(109.5)) = f(99.5) = f(f(110.5)) = f(100.5)
\end{aligned}$$

But we stop one step earlier, since now we are larger than 100, so $f(100.5) = 90.5$.

Finally, we have $f(30\pi)$. Before we begin, note that $30\pi + 6 > 100$.

$$\begin{aligned}
f(30\pi) &= f(f(30\pi + 11)) = f(30\pi + 1) = f(f(30\pi + 12)) = f(30\pi + 2) = f(f(30\pi + 13)) \\
&= f(30\pi + 3) = f(f(30\pi + 14)) = f(30\pi + 4) = f(f(30\pi + 15)) = f(30\pi + 5) \\
&= f(f(30\pi + 16)) = f(30\pi + 6) = 30\pi - 4
\end{aligned}$$

Since $30\pi + 6 > 100$.

b) Compute several more values of $f(x)$ for both integer and non-integer values of x .

Once you are satisfied that you understand how the function works, sketch an anatomically correct graph on the domain of $[90, 110]$. Explain why the graph must look as you claim it does.

It is clear that for all integers k less than 100, $f(k) = 91$ and for all real numbers greater than 100, $f(k) = k - 10$. The issue is what happens for non-integer values less than 100.

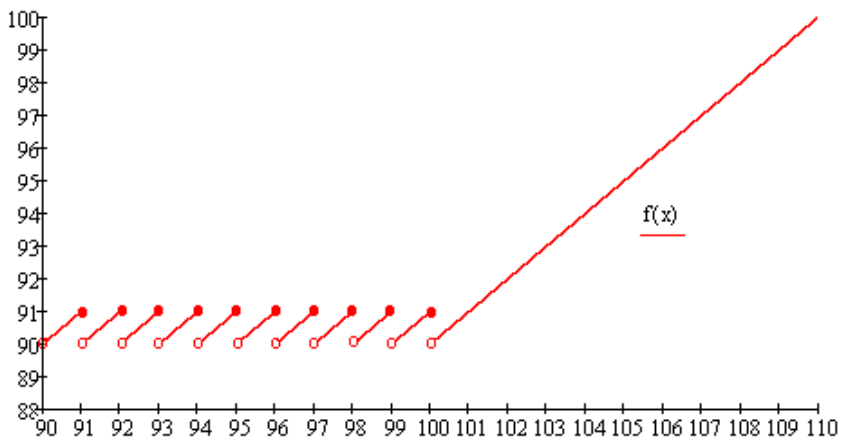
Consider $f(k + \varepsilon)$ where k is an integer less than 100 and ε is sufficiently small, so $k + \varepsilon$ is less than $k + 1$.

$$f(k + \varepsilon) = f(f(k + 11 + \varepsilon)) = f(k + 1 + \varepsilon) = f(f(k + 12 + \varepsilon)) = f(k + 2 + \varepsilon) = \dots = f(k + n + \varepsilon)$$

and we continue until, we reach a point where $k + n = 100$ so that $k + n + \varepsilon > 100$.

This, then, becomes
 $f(100 + \varepsilon) = 90 + \varepsilon$. So,
 $f(k) = 91$, $f(k + 1) = 91$,
and $f(k + \varepsilon) = 90 + \varepsilon$.

This gives the graph shown at right.



c) By appealing to your graph or otherwise, find $f'(x)$ and clearly indicate the domain of the derivative.

Since this function is piece-wise linear and all slopes are 1, the derivative is the constant function $f'(x) = 1$ for all except the integers less than 101. The derivative is not defined for those values since the function is not continuous.

d) Let $g(x) = f(f(x))$ and let $h(x) = f(x-10)$. Find a relationship between the two functions g and h . Explain why you believe this is the correct relationship.

I thought this was really hard until I finally figured out what was going on.

If we look at

$$\begin{aligned} g(98) &= f(f(98)) = f(f(f(109))) = f(f(99)) = f(f(f(110))) = f(f(100)) = f(f(f(111))) \\ &= f(f(101)) = f(91) = 91. \end{aligned}$$

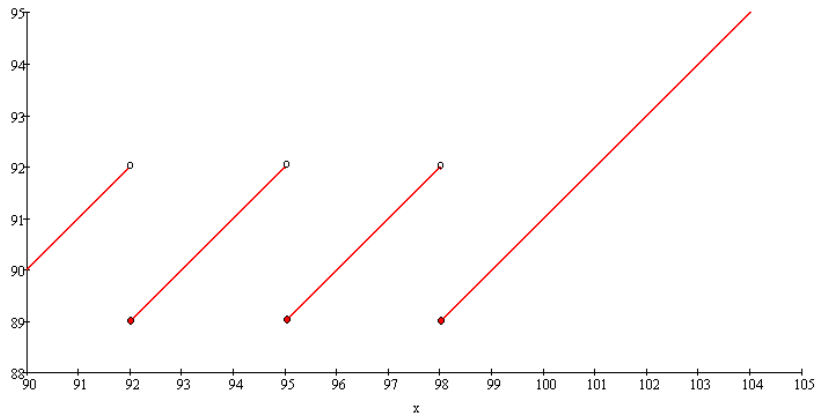
No change from before. But suppose we try $g(108)$. We know that $f(108) = 98$, but $g(108) = f(f(108)) = f(98) = 91$ and $g(110) = f(f(110)) = f(100) = 91$ and $g(111) = f(f(111)) = f(101) = 91$. But $g(112) = f(f(112)) = f(192) = 92$.

In this case, we have to have $f(k) > 100$ to exit the recursion, and this means $k > 110$. So the graph of $g(x) = f(f(x))$ is the graph of f shifted to the right by 10 units. That is $g(x) = f(x-10)$. So $g(x) = h(x)$.

e) Let $g(x) = f(f(x))$. Find $g'(x)$. Clearly indicate the domain of $g'(x)$.

Since $g(x) = f(f(x))$ is another representation of $g(x) = f(x-10)$, the derivative of g is $f'(x-10)$, so $g'(x) = 1$ for all except the integers less than 111, where the derivative is not defined.

f) Find values of a , b , and c so that $F(x) = \begin{cases} x-a & \text{if } x \geq b \\ F(F(x+c)) & \text{if } x < b \end{cases}$ has the graph shown below:



First, note the effect of changing the first inequality from $x-a$ if $x > b$ to $x-a$ if $x \geq b$. The points of definition at the discontinuities is now on the lower end.

The difference $c - a$ determines the length of the segments while the smallest value on the segments has the value $b - a$. We need the difference in c and a to be 3, since the segments cover an interval 3 units long and 3 units high. These segments end at $(b, b - a)$. In this case we have $b = 98$ and $b - a = 89$, so $a = 9$. Since $c - a = 3$, we know that $c = 12$. The line segments are bounded between $y = b - a$ and $y = b - 2a + c$.

