Consider the recursively defined function \( f(x) = \begin{cases} x - 10 & \text{if } x > 100 \\ f(f(x + 11)) & \text{if } x \leq 100 \end{cases} \).

a) Find the values of \( f(92) \), \( f(85) \), \( f(92.5) \) and \( f(30\pi) \).

This is a bit tedious until you see what is happening. Using the definition, we have

\[
\begin{align*}
f(92) &= f(f(103)) = f(93) = f(f(104)) = f(94) = f(f(105)) = f(95) = f(f(106)) = f(96) \\
&= f(f(107)) = f(97) = f(f(108)) = f(98) = f(f(109)) = f(99) = f(f(110)) = f(100) \\
&= f(f(111)) = f(101) = 91
\end{align*}
\]

So, \( f(92) = 91 \) as is also \( f(93), f(94), f(95), f(96), f(97), f(98), f(99), f(100), \) and \( f(101) \).

Similarly, we have

\[
\begin{align*}
f(85) &= f(f(95)) = f(f(f(106))) = f(f(96)) = f(f(f(107))) = f(f(97)) = f(f(f(108))) \\
&= f(f(98)) = f(f(f(109))) = f(f(99)) = f(f(f(110))) = f(f(100)) = f(f(f(111))) \\
&= f(f(101)) = f(f(91)) = f(f(f(102))) = f(f(92))
\end{align*}
\]

Aha! We know that \( f(92) = 91 \), so \( f(f(92)) = f(91) \) and \( f(91) = f(f(102)) = f(92) = 91 \).

So, \( f(85) = 91 \).

Now, what about non-integers. We can use the string generated by \( f(92) \)
\[ f(92.5) = f(f(103.5)) = f(93.5) = f(f(104.5)) = f(94.5) = f(f(105.5)) = f(95.5) \]
\[ = f(f(106.5)) = f(96.5) = f(f(107.5)) = f(97.5) = f(f(108.5)) = f(98.5) \]
\[ = f(f(109.5)) = f(99.5) = f(f(110.5)) = f(100.5) \]

But we stop one step earlier, since now we are larger than 100, so \( f(100.5) = 90.5 \).

Finally, we have \( f(30\pi) \). Before we begin, note that \( 30\pi + 6 > 100 \).

\[ f(30\pi) = f(f(30\pi + 11)) = f(30\pi + 1) = f(f(30\pi + 12)) = f(30\pi + 2) = f(f(30\pi + 13)) \]
\[ = f(30\pi + 3) = f(f(30\pi + 14)) = f(30\pi + 4) = f(f(30\pi + 15)) = f(30\pi + 5) \]
\[ = f(f(30\pi + 16)) = f(30\pi + 6) = 30\pi - 4 \]

Since \( 30\pi + 6 > 100 \).

b) Compute several more values of \( f(x) \) for both integer and non-integer values of \( x \).

Once you are satisfied that you understand how the function works, sketch an anatomically correct graph on the domain of \([90, 110]\). Explain why the graph must look as you claim it does.

It is clear that for all integers \( k \) less than 100, \( f(k) = 91 \) and for all real numbers greater than 100, \( f(k) = k - 10 \). The issue is what happens for non-integer values less than 100.

Consider \( f(k + \varepsilon) \) where \( k \) is an integer less than 100 and \( \varepsilon \) is sufficiently small, so \( k + \varepsilon \) is less than \( k + 1 \).

\[ f(k + \varepsilon) = f(f(k + 11 + \varepsilon)) = f(k + 1 + \varepsilon) = f(f(k + 12 + \varepsilon)) = f(k + 2 + \varepsilon) = \cdots = f(k + n + \varepsilon) \]

and we continue until, we reach a point where \( k + n = 100 \) so that \( k + n + \varepsilon > 100 \).

This, then, becomes \( f(100 + \varepsilon) = 90 + \varepsilon \). So,

\( f(k) = 91, f(k + 1) = 91, \)

and \( f(k + \varepsilon) = 90 + \varepsilon. \)

This gives the graph shown at right.
c) By appealing to your graph or otherwise, find \( f'(x) \) and clearly indicate the domain of the derivative.

Since this function is piece-wise linear and all slopes are 1, the derivative is the constant function \( f'(x) = 1 \) for all except the integers less than 101. The derivative is not defined for those values since the function is not continuous.

d) Let \( g(x) = f(f(x)) \) and let \( h(x) = f(x - 10) \). Find a relationship between the two functions \( g \) and \( h \). Explain why you believe this is the correct relationship.

I thought this was really hard until I finally figured out what was going on.

If we look at
\[
g(98) = f(f(98)) = f(f(f(109))) = f(f(99)) = f(f(f(110))) = f(f(100)) = f(f(f(111)))
\]
\[
= f(f(101)) = f(91) = 91.
\]

No change from before. But suppose we try \( g(108) \). We know that \( f(108) = 98 \), but
\[
g(108) = f(f(108)) = f(98) = 91 \text{ and } g(110) = f(f(110)) = f(100) = 91 \text{ and}
\]
\[
g(111) = f(f(111)) = f(101) = 91. \text{ But } g(112) = f(f(112)) = f(192) = 92.
\]

In this case, we have to have \( f(k) > 100 \) to exit the recursion, and this means \( k > 110 \). So the graph of \( g(x) = f(f(x)) \) is the graph of \( f \) shifted to the right by 10 units. That is \( g(x) = f(x - 10) \). So \( g(x) = h(x) \).

e) Let \( g(x) = f(f(x)) \). Find \( g'(x) \). Clearly indicate the domain of \( g'(x) \).

Since \( g(x) = f(f(x)) \) is another representation of \( g(x) = f(x - 10) \), the derivative of \( g \) is \( f'(x - 10) \), so \( g'(x) = 1 \) for all except the integers less than 111, where the derivative is not defined.
f) Find values of $a$, $b$, and $c$ so that $F(x) = \begin{cases} x - a & \text{if } x > b \\ F(F(x + c)) & \text{if } x \leq b \end{cases}$ has the graph shown below:

First, note the effect of changing the first inequality from $x - a$ if $x > b$ to $x - a$ if $x \geq b$. The points of definition at the discontinuities is now on the lower end.

The difference $c - a$ determines the length of the segments while the smallest value on the segments has the value $b - a$. We need the difference in $c$ and $a$ to be 3, since the segments cover an interval 3 units long and 3 units high. These segments end at $(b, b - a)$. In this case we have $b = 98$ and $b - a = 89$, so $a = 9$. Since $c - a = 3$, we know that $c = 12$. The line segments are bounded between $y = b - a$ and $y = b - 2a + c$. 