Any Way You Slice It

Making a sandwich for the twins to share is always a problem. Lucy always insists on having equal amounts of bread, while Linus insists on have equal amounts of crust. They both insist that the two parts be cut on a diagonal (as shown at right). Is it possible to satisfy both of the twins?

For our model, we will consider a slice of bread to be a rectangle surmounted by a semi-ellipse. The semi-ellipse has the equation $y = \frac{b}{a} \sqrt{a^2 - x^2}$. The bread is to be cut from the corner at $(-a, -h)$ to point $C$, with coordinates $\left( c, \frac{b}{a} \sqrt{a^2 - c^2} \right)$. We will need to find the value of $c$, the $x$-coordinate of point $C$, in terms of $a$, $b$, and $h$.

1) First, let’s consider Lucy’s request for equal amounts of bread. The area of an ellipse with major axis $2M$ and minor axis $2m$ is $A_{\text{ellipse}} = \pi (M \cdot m)$. Note, a circle is the special case when $M = m$.

Find the area for this slice of bread and, consequently, the area for half a slice.

The bread is a $2a \times h$ rectangle and half an ellipse with major axis $2a$ and minor axis $2b$, so the total area is $(2ah) + \left( \frac{ab\pi}{2} \right)$. So, half of this area is $ah + \frac{\pi ab}{4}$.

2) Explain why the equation

$$\frac{\pi ab}{4} + ah = \int_{-a}^{c} \left( \frac{b}{a} \sqrt{a^2 - x^2} - \frac{b}{a} \sqrt{a^2 - c^2} + \frac{b}{a} \frac{\sqrt{a^2 - c^2}}{c + a} (x + a) + h \right) dx$$

contains the solution to Lucy’s request. What is being calculated in this integral equation?

The line from the corner $(-a, -h)$ to the point $C$ is given by $f(x) = \frac{b}{a} \sqrt{a^2 - x^2} + \frac{b}{a} \frac{\sqrt{a^2 - c^2}}{c + a} (x + a) - h$. So, the left side of the equation is the area of half the bread, and the right side is the area between the ellipse and the line cutting the bread. This is the area shaded in yellow in the figure. Finding the value of $c$ for which the two areas are equal (half the slice) is the goal.

3) Perform the integration in 2). In your solution, approximate $Sin^{-1}\left( \frac{c}{a} \right)$ with $\frac{c}{a}$, and solve the resulting equation for $c$.

Let’s do this in pieces.
\[ \int_{-a}^{c} \left( b \sqrt{a^2 - x^2} - \frac{b}{a} \sqrt{a^2 - c^2 + h} (x + a) + h \right) \, dx = \int_{-a}^{c} \left( b \sqrt{a^2 - x^2} \right) \, dx - \int_{-a}^{c} \left( \frac{b}{a} \sqrt{a^2 - c^2 + h} (x + a) + h \right) \, dx. \]

The second integral is easy, it is just, so it is the first integral that is tough.

\[ \int_{-a}^{c} \left( \frac{b}{a} \sqrt{a^2 - c^2 + h} (x + a) + h \right) \, dx = \frac{b}{a} \sqrt{a^2 - c^2 + h} \left( \frac{(x + a)^2}{2} \right) + hx \bigg|_{-a}^{c} = \left( c + a \right) \left( h - \frac{b}{a} \sqrt{a^2 - c^2} \right). \]

The first integral requires a trig substitution \( x = a \sin \theta, \ x^2 = a^2 \sin^2 \theta, \) and \( dx = a \cos \theta \, d\theta \) or we can look it up in an integral table.

\[ \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right), \] so

\[ \int_{-a}^{c} \left( \frac{b}{a} \sqrt{a^2 - x^2} \right) \, dx = \frac{b}{a} \left( \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) \right) \bigg|_{-a}^{c} = \frac{b}{a} \left( \frac{c}{2} \sqrt{a^2 - c^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{c}{a} \right) \right) - \frac{b}{a} \left( 0 + \frac{a^2}{2} \sin^{-1} (1) \right) \]

which simplifies to

\[ \frac{bc}{2a} \sqrt{a^2 - c^2} + \frac{ba}{2} \sin^{-1} \left( \frac{c}{a} \right) - \frac{ba \pi}{4}. \]

Finally, we have

\[ \frac{\pi ab}{4} + ah = \left( \frac{bc}{2a} \sqrt{a^2 - c^2} + \frac{ba}{2} \sin^{-1} \left( \frac{c}{a} \right) \right) - \left( \frac{ba \pi}{4} \right) + \left( \frac{c + a}{2} \right) \left( h - \frac{b}{a} \sqrt{a^2 - c^2} \right) \]

to solve for \( c \) with the substitution \( \frac{c}{a} = \sin^{-1} \left( \frac{c}{a} \right) \). Ouch! It’s “just” as quadratic, but it is a real mess.

Clearly, the \( \frac{ab \pi}{4} \)'s and \( \frac{bc}{2a} \sqrt{a^2 - c^2} \) add out and we can simplify the \( ah \) terms and multiply the mess by 2. That helps a lot.

This leaves \( h(c - a) + b\sqrt{a^2 - c^2} - bc = 0 \). Now we can write \( b\sqrt{a^2 - c^2} = bc + h(a - c) \) and \( b\sqrt{a^2 - c^2} = (b - h)c + ha \). Square both sides to find \( b^2 \left( a^2 - c^2 \right) = (b - h)^2 c^2 + 2ha(b - h)c + h^2 a^2 \).

Now \( b^2 a^2 - bc^2 = (b - h)^2 c^2 + 2ha(b - h)c + h^2 a^2 \). Multiply it all out and use the quadratic formula.

In the end, we have \( c = \frac{ah(b + h) \pm \sqrt{b^2(b + h)}}{b^2 + (b + h)^2} \). There are about 1,000 ways to write this depending on how you group your terms.
4) Using a real slice of your own bread (include photo), or alternatively, the figure at right, estimate the values of $a$, $h$, and $b$, and find the coordinates of point $C$.

I blew up the bread so that the numbers were nice. I estimated $a = 35$, $h = 45$, and $b = 30$ millimeters.

So, $c = \frac{ah(b + h) \pm b\sqrt{2b(b + h)}}{b^2 + (b + h)^2} = 28.298$.

The coordinates of point $C$ are $(28.898, 17.654)$.

To handle Linus’ request, we need to use the parametric form of the ellipse, $x = a \cos \theta$, $y = b \sin \theta$.

5) Write an integral equation whose value is the perimeter of the entire slice of bread.

The perimeter is the three sides of the rectangle, $2h + 2a$, and the boundary of the ellipse

$$
\int_0^\pi \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta,
$$

so $L = 2a + 2h + \int_0^\pi \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta$.

6) Explain why the length of the crust of the unshaded region in Figure 1 can be found by solving for $c$ in the equation

$$
2a + h + \int_0^{\arccos(c/a)} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta = h + a + \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta.
$$

The right side of the equation is half of the perimeter and the left side is the perimeter of the unshaded region, since it includes the bottom ($2a$), one side ($h$), and the portion of the area up to point $C$. Finding the value of $c$, that satisfies this equation will give you half of the perimeter.

7) The equation above cannot be solved analytically, but we can solve it numerically as well as in some special cases.

(a) Suppose the top is really a semi-circle, so $a = b$. Find the value of $c$ that equalizes the crust between Linus and Lucy.
If $a = b$, then our equation becomes:

$$2a + h + \int_0^{\cos^{-1}(c/a)} a \, d\theta = h + a + \int_0^{\pi/2} a \, d\theta$$
or

$$1 + \int_0^{\cos^{-1}(c/a)} d\theta = \int_0^{\pi/2} d\theta.$$

So, $\cos^{-1}(c/a) = \frac{\pi}{2} - 1$ and

$$c = a \cos \left( \frac{\pi}{2} - 1 \right) = a \cos \frac{\pi}{2} \cos 1 + a \sin \frac{\pi}{2} \sin 1 = a \sin 1 \approx 29.45 \text{ mm}.$$

(b) Suppose $b = 0$, is the value of $c$ what you would expect?

If $b = 0$, then

$$1 + \int_0^{\cos^{-1}(c/a)} \sin \theta \, d\theta = \int_0^{\pi/2} \sin \theta \, d\theta.$$

This is equivalent to

$$1 - \cos \left( \cos^{-1} \frac{c}{a} \right) + \cos 0 = -\cos \frac{\pi}{2} + \cos 0 \text{ or } 1 - \frac{c}{a} = 0,$$

so $c = a$ and we have a 45 degree cut along the diagonal as expected.

(c) With the dimension of your own bread or the slice in Figure 2, numerically solve the equation above for $c$.

We need to solve

$$a + \int_0^{\cos^{-1}(c/a)} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta - \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta = 0$$

with $a = 35$, $h = 45$, and $b = 30$ millimeters. I found $c = 30.212$ mm.

8) How close can you come to satisfying both Luc and Linus? That is, how far apart are the locations for $C$ that equalize the crust and the amount of bread?

For even area, we have $c = 28.9$ mm and for equal crust we have $c = 30.2$, so using $c = 30$ mm comes close to satisfying both.