

NCAAPMT Calculus Challenge 2011-2012

Challenge #3

Due: October 26, 2011

A Model of Traffic Flow

Everyone has at some time been on a multi-lane highway and encountered road construction that required the traffic to occupy only one lane each way. Naturally, the Department of Transportation would like to maximize the flow of traffic through this stretch of the highway. What speed limit should be set for such a stretch of road to ensure the greatest traffic flow while also maintaining safety?

When developing a mathematical model of a real-world situation, it is usually necessary to make some simplifying assumptions. In this model, we assume that all the cars are the same length L and that the cars follow each other at a distance d (see Figure 1). We know from experience that the faster we drive, the more distance we should leave between our car and the car in front of us. Therefore, we want our model to reflect the fact that the following distance d depends upon the assigned speed limit s .



Figure 1: Diagram of the flow of cars on the highway

What is meant by traffic flow? Traffic flow is the measure of the number of cars that pass a given point in a given unit of time. The units for traffic flow is cars per unit time (cars per second, cars per hour, etc.). Imagine the situation described in the paragraph above. Cars are flowing uniformly down the road, each traveling at speed s and leaving a distance of d between them and the car in front. How can we develop an equation to model of the flow of traffic from these basic assumptions?

Developing the Model

Imagine that you are standing beside the road watching the cars pass you by. If a car has just passed you, when will the next car pass? The time t required for the next car to pass you is the ratio of the distance it has to travel and the speed at which it is moving, that is

$$t = \frac{L+d}{s}. \quad (1)$$

This equation represents the time interval that elapses between the passage of one car and the next beside a fixed point on the road. The implied units are seconds per car. . The reciprocal of this expression, then, represents traffic flow: the number of cars that pass a fixed point per unit time. (For example, if 5 seconds elapse between two cars passing a point, then the traffic flow

must be 1/5 of a car per second—or equivalently, 12 cars per minute.) Thus, one simple model of traffic would be the equation

$$F = \frac{s}{L + d}. \quad (2)$$

We note now that while highway speed limits are typically given in miles per hour in the U.S., car lengths are not generally estimated in miles, but are given in feet. So a unit change is in order. Also, we all know that the following distance d , depends on the speed of the cars.

Three different “rules of thumb” are commonly used to determine a safe following distance. You might want to check your state’s Driver’s Manual to see what is recommended. The faster you are going the greater the distance must be between you and the car in front of you to give you time and distance to safely stop if the car in front of you stops suddenly.

- a) *Follow two car lengths for every 10 mph.*
- b) *Follow three seconds behind the car in front.*
- c) *The distance needed to stop varies at different speeds and includes thinking and braking distances. A table of stopping distances is often given.*

Table 1 is from a driver’s handbook and gives some typical distances.

Speed (mph)	Thinking Distance (ft)	Braking Distance (ft)	Stopping Distance (ft)
20	20	20	40
30	30	45	75
40	40	80	120
50	50	125	175
60	60	180	240

1. Modify equation (2) by replacing d with a function of s based on each of the three “rules of thumb”. For each, determine the speed that maximizes the flow of traffic in terms of the car length L .
2. Select the model that best describes traffic flow and which has a maximum flow. Since this model is dependent on the length of the vehicle, the maximum traffic flow will also be dependent on the length. Graph traffic flow vs. speed and identify the ordered pair that represents maximum traffic flow for each of the following vehicles:
 - a. *Mini Cooper has length of 142.8 inches.*
 - b. *Toyota Corolla has length of 178.3 inches.*
 - c. *Hummer H2 has length of 189.9 inches.*
 - d. *An eighteen-wheeler with average length between 70 and 80 feet*

3. The web site

http://www.thetruckersreport.com/truck_facts.shtml#Length%20of%20eighteen%20wheelers

gives the statement that “The length of time to stop an eighteen wheeler is 40% greater than that of an automobile.” Apply this information to find the ordered pair that represents the speed for maximum traffic flow for an eighteen-wheeler.

4. Our experience suggests that drivers do not leave as much space between cars as they should. Let p be the fraction of the required stopping distance that the drivers actually leave between cars. Assume $0.1 \leq p \leq 1$. Find the optimal speed and its corresponding traffic flow that maximizes traffic flow in terms of the parameters L and p . Which has a larger effect of the optimal speed, L or p ?

5. Some might argue that the drivers would never drive so close to the car in front of them that they would not have sufficient time to react. These drivers might (unconsciously, of course) leave all of the reaction distance but only a fraction of the braking distance. Adjust your model by multiplying braking distance by p , where $0.1 \leq p \leq 1$ to find the optimal speed and its corresponding traffic flow for the Corolla for $p = 0.1$, $p = 0.5$, and $p = 1$.