Could You Study World History in Japan?

This is the final challenge for this year. I hope you have enjoyed working on them. Most of you will be off to college next year. As a part of the process, you have taken either the SAT or the ACT. Many students take the AP Calculus exam as a way of strengthening their college applications.

But suppose your AP exam determined whether you would be admitted or not? Having your college admission riding on your AP score would add a level of difficulty to the problems, since so much depends on doing well.

This final challenge consists of three problems from a Japanese university’s entrance exams. In order to be admitted to the university, students must be able to solve problems in geometry, algebra, and calculus. This is true regardless of their major, although the problems are more challenging for STEM majors. These three problems are not for science, mathematics, or engineering students, but for students interested in a degree in the social science (history, languages, psychology, etc). Could you gain entrance to a humanities discipline in a Japanese University?

1. Given the cubic function \( y = x^3 + 3t^2x + t^3 \).
   (a) Consider \( y \) as a function of \( t \), with parameter \( x > 0 \). Find the relative extrema of \( f(t) \).
   (b) Determine the region in the \( xy \)-plane for which \( y = x^3 + 3t^2x + t^3 \) is satisfied for some value of \( t > 0 \). That is, in the \( xy \)-coordinate plane, shade in all the points which lie on the graph of \( y = x^3 + 3t^2x + t^3 \) for some \( t \).

2. Define the two functions \( f_1(x) = x^2 - 2|x| + 1 \) and \( f_2(x) = x^2 - 2|x| + (1-k^2) \), with \( 0 < k < 1 \).
   (a) Find \( A_1 \), the finite area, centered on the \( y \)-axis, bounded by \( y = f_1(x) \) and the \( x \)-axis.
   (b) Find \( A_2 \) in terms of \( k \), where \( A_2 \) is the finite area, centered on the \( y \)-axis, bounded by the function \( y = f_2(x) \) and the \( x \)-axis.
   (c) For what values of \( k \) is \( A_2 \) larger than \( A_1 \)?
3. Consider the three points in the xy-plane, \( A(-1,0) \), \( B(1,0) \), and \( P(t, 2t^2 + 1) \). The angle bisector of \( \angle APB \) intersects the \( x \)-axis at point \( Q \). Define a function by \( f(t) = \frac{QB}{AQ} \), where \( QB \) and \( AQ \) are the lengths of the segments \( QB \) and \( AQ \), respectively. Find the maximum and minimum value of \( f(t) \).