

NCAAPMT Calculus Challenge 2011-2012
Challenge #4 **SOLUTION (finally!)**

A Variation on the Classic Wire Problem

You have all probably done “the wire problem”. Take a wire of length L , (a pipe cleaner works well), cut it at some point along its length. Use one section to make a square and the other section to make a circle. Find the point at which the wire should be cut to maximize the total area of the circle and square. Calculus teachers like to give this problem since the optimal location is at one of the endpoints of the domain. This reinforces the need to not only check critical values of the objective function, but also consider the endpoints of its domain.

We are going to look at the same problem, but instead focus on finding the location for the cut that *minimizes* the total area of the two figures.



1) How much wire should be used for the circle and how much for the square if we wish to *minimize* the total area of the two regions so defined? Prove that the area is minimal.

As shown in the diagram above, let the circumference of the circle be x , and the perimeter of the square be $L-x$. Then, $2\pi r = x$, so $r = \frac{x}{2\pi}$ and the area of the circle is $A_c = \frac{x^2}{4\pi}$. The side length

of the square is $s = \frac{L-x}{4}$, and the area of the square is $A_s = \frac{(L-x)^2}{16}$. The total area of the two

figures is $A = A_s + A_c = \frac{(L-x)^2}{16} + \frac{x^2}{4\pi}$.

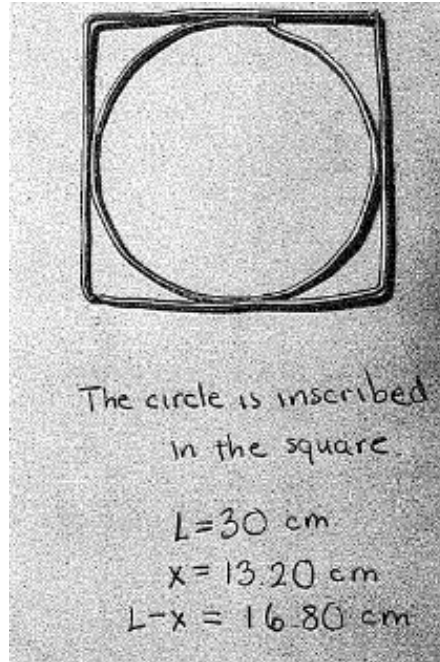
Now, for the calculus.

$\frac{dA}{dx} = \frac{(L-x)}{8} + \frac{x}{2\pi} = \frac{2\pi L - (2\pi + 8)x}{8\pi}$. If $\frac{dA}{dx} = 0$ then $x = \frac{\pi L}{\pi + 4}$, and the area is

$$A = A_s + A_c = \frac{\left(L - \frac{\pi L}{\pi + 4}\right)^2}{16} + \frac{\pi^2 L^2}{4\pi(\pi + 4)^2} = \frac{L^2}{4(\pi + 4)}.$$

Since $\frac{d^2A}{dx^2} = \frac{1}{2\pi} > 0$ always, this is a minimum value for the combined areas.

2) Now, with a pipe cleaner or a coat hanger, actually make the circle and square with dimensions given by your solution to 1). Put the two figures together (take a photo and submit it with you solution if you want). What is the relationship between these two figures? Prove that your observation is, in fact, correct.



Picture courtesy of Farmington High

We know that $r = \frac{x}{2\pi}$, $s = \frac{L-x}{4}$, and $x = \frac{\pi L}{\pi+4}$. So,

$$r = \frac{xL}{2(4+\pi)} \text{ and } s = \frac{L}{4+\pi}, \text{ which means that } 2r = s,$$

and the diameter of the circle is the same as the side of the square. The circle can be precisely inscribed in the square.

3) Is this just a coincidence? **(NO!)** Suppose instead of a circle and a square, you find the length of wire needed to bend the wire into a circle and an equilateral triangle which minimizes the combined area. Does the relationship found in 2) still exist? Prove this result.

Since the area of the polygon will be more difficult to describe, we will let the length x be bent into the polygon and $L-x$ into the circle. In this case, we have $r = \frac{L-x}{2\pi}$ and $s = \frac{x}{3}$. Now, the area

of the circle is $A_c = \frac{(L-x)^2}{4\pi}$ and the area of the triangle is $A_t = \frac{s^2\sqrt{3}}{4} = \frac{x^2\sqrt{3}}{36}$.

The total area is $A(x) = \frac{x^2\sqrt{3}}{36} + \frac{(L-x)^2}{4\pi}$, so $A'(x) = \frac{x\sqrt{3}}{18} - \frac{(L-x)}{2\pi}$. Solving, we find

$x = \frac{9L}{\pi\sqrt{3}+9}$. The second derivative is always positive, so this is the location of the minimum

value. Since $x = \frac{9L}{\pi\sqrt{3}+9}$, $r = \frac{\sqrt{3}}{2} \left(\frac{L}{9+\pi\sqrt{3}} \right)$ and $s = \frac{3L}{9+\pi\sqrt{3}}$. We note that $r = \left(\frac{1}{3} \right) \frac{s\sqrt{3}}{2}$, so

the circle is precisely inscribed in the triangle.

4) Suppose instead you find the length of wire needed so that a square and an equilateral triangle are formed which minimize the combined area. Is the square inscribed in the triangle or the triangle inscribed in the square (or neither)? Prove this result.

In this case, the one figure does not circumscribe the other. This seems to require a property of circles.

The mathematical details: the side of the triangle is t , and the side of the square is s . So, $t = \frac{x}{3}$ and

$s = \frac{L-x}{4}$, so $A(t) = \frac{x^2\sqrt{3}}{36} + \frac{(L-x)^2}{16}$. Differentiating, we have $A'(t) = \frac{x\sqrt{3}}{18} - \frac{(L-x)}{8}$. Solving

for $A'(x) = 0$, we find $x = \frac{9}{9+4\sqrt{3}}$. Then $t = \frac{3}{9+4\sqrt{3}}$ and $s = \frac{\sqrt{3}}{9+4\sqrt{3}}$. These dimensions do not fit together.

5). Repeat 3) with a circle and either a pentagon, hexagon, or octagon. What did you find?

I used a hexagon, since it has the nice geometry of six equilateral triangles. In this case, $s = \frac{x}{6}$ and

$r = \frac{L-x}{2\pi}$, so the total area is $A(t) = \frac{3\sqrt{3}}{2} \left(\frac{x^2}{36} \right) + \frac{(L-x)^2}{4\pi}$. With $A'(t) = \frac{\sqrt{3}x}{12} - \frac{(L-x)}{2\pi}$, we find

that $x = \frac{6L}{\pi\sqrt{3}+6}$ minimizes the area. This gives $s = \frac{L}{\pi\sqrt{3}+6}$ and $r = \frac{L\sqrt{3}}{2(\pi\sqrt{3}+6)}$. Note that

$r = \frac{\sqrt{3}}{2}s$, so r is the altitude for each of those equilateral triangles, which means the circle can be inscribed in the hexagon. Other regular polygons work in the same way.

6) As a thought experiment (this means, don't actually do the computation), suppose you find the cut point so that a 7-gon and a 100-gon are created with minimum total area. If you created the polygons and put the two together, what would you see?

Since the 100-gon will be essentially indistinguishable from a circle, I would expect that it would appear that the 7-gon circumscribes the 100-gon, although there will be some small overlap.

Super Challenge: The area of a regular n -gon of side length s is $A = \frac{s^2}{4} \cdot n \cdot \cot\left(\frac{\pi}{n}\right)$. Prove that

the n -gon and circle always have the relationship found in 2).

Hickman High School has a fantastic presentation of this result, which I'll present next week. I want to get as caught up as I can with the other problems before I type that up. But it is a beautiful result.