

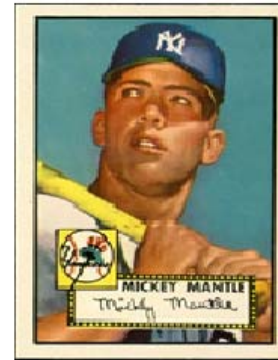
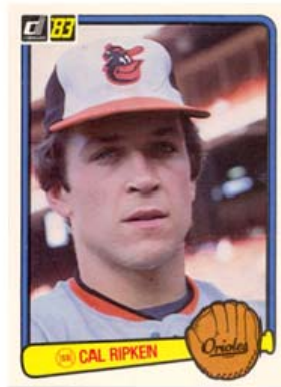
NCAAPMT Calculus Challenge 2011-2012

Challenge #5

SOLUTION

Since the Thanksgiving Holidays are coming up, this project is due after that break.

When to Sell the Baseball Card?



A dealer in sports cards has a rare baseball card, and he's trying to decide when to sell it. He knows its value will grow over time, but he could sell the card and invest the money in a bank account, and the value of the money would also grow over time due to interest. The question is: when should the dealer sell the card?

Experience suggests to the dealer that over time, the value of the card, like many collectibles, will grow in a way consistent with the following model:

$$V(t) = Ce^{k\sqrt{t}},$$

where $C > 0$ and $k > 0$ are constants, V is the value of the baseball card in dollars, and t is the number of years after the present time.

1. Does $V(t)$ have a maximum value for some t ? If so, interpret this value. If not, explain how you know a maximum does not exist.

If, $V(t) = Ce^{k\sqrt{t}}$, then $\frac{dV}{dt} = \frac{Cke^{k\sqrt{t}}}{2\sqrt{t}}$. We see that $\frac{dV}{dt} \neq 0$ for any time t , and $\frac{dV}{dt}$ does not exist only at $t = 0$,

where the value is $V(0) = C$. Since $\frac{dV}{dt} > 0$ thereafter, this is the minimum value for the card.

2. Does $V(t)$ have a point of inflection for some t ? If so, interpret this value. If not, explain how you know a point of inflection does not exist.

Since $\frac{dV}{dt} = \frac{Cke^{k\sqrt{t}}}{2\sqrt{t}}$, we have $\frac{d^2V}{dt^2} = \frac{\sqrt{t}Ck^2e^{k\sqrt{t}} - Cke^{k\sqrt{t}}}{4t\sqrt{t}} = \frac{Cke^{k\sqrt{t}}(k\sqrt{t} - 1)}{4t\sqrt{t}}$. If this is zero, then $t = \frac{1}{k^2}$. All

other terms in the second derivative are positive, so if $t < \frac{1}{k^2}$, $\frac{d^2V}{dt^2} < 0$ then the curve is concave down and if

$t > \frac{1}{k^2}$, $\frac{d^2V}{dt^2} > 0$, the curve is concave up. So there is a point of inflection.

This point of inflection is the time at which the value of the card begins to take off. For the first several years, the card slowly increases in value, but it hasn't yet become a "find". Once it reaches a certain age, $t = \frac{1}{k^2}$, it is now a vintage card and its price booms.

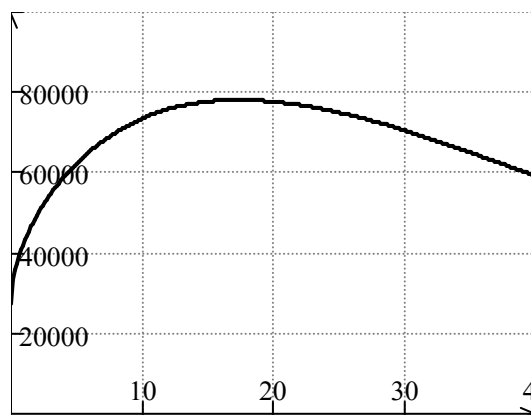
Suppose that the dealer, who is 18 years old, decides to sell the card at time t , sometime in the next 42 years: $0 \leq t \leq 42$. At that time t , he'll invest the money he gets for the sale of the card in a bank account that earns an interest rate of r , compounded continuously. When he turns 60, he'll take the money that's in his bank account and begin to draw on it for his retirement. Let A be the amount of money in his account when he turns 60. (There was a typo in this part of the problem. I said 42 years but put retirement at age 65. Well, they say there are three kinds of mathematicians: those who can count and those who can't. The result isn't affected by the difference in $42 - t$ and $47 - t$, since, as you will see, that term never plays a role in the result. Either way you worked it is OK. I apologize for any confusion.)

3. Explain why the amount of money in the dealer's account when he turns 60 can be modeled by

$$A(t) = \left(Ce^{k\sqrt{t}} \right) e^{r(42-t)}.$$

The value of amount the dealer receives for the card at retirement comes from two components. The first is the actual value of the card, given by $V(t) = \left(Ce^{k\sqrt{t}} \right)$. This value represents the amount of money the dealer puts into the bank to earn interest. That initial installment earn interest for the remaining time, $(42 - t)$ years. So, the earned interest is $A(t) = \left(Ce^{k\sqrt{t}} \right) e^{r(42-t)}$.

4. Let $C = 2500$, $k = 0.5$, and $r = 0.06$. When should the dealer sell the card so as to maximize the amount in his retirement account when he turns 60?



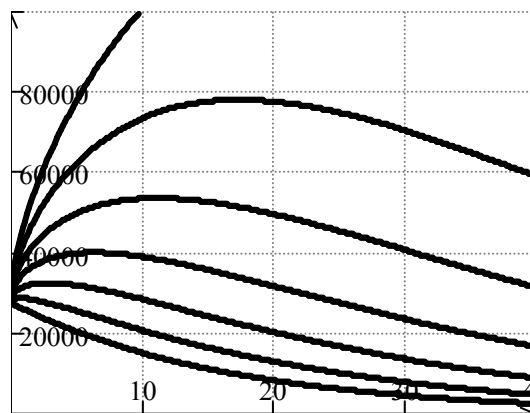
We have $A(t) = \left(2500e^{0.5\sqrt{t}} \right) e^{0.06(42-t)}$, so

$2500 \left(e^{0.5\sqrt{t} + 0.06(42-t)} \right) \left(\frac{0.05}{2\sqrt{t}} - 0.06 \right)$. Since the exponential is never zero, we have only $\frac{0.05}{2\sqrt{t}} - 0.06 = 0$ to

consider. Solving, we find that $\frac{0.05}{2\sqrt{t}} - 0.06 = 0 \rightarrow t = 17.36$ years. The dealer should hold the card for a little over 17 years, sell it, and invest the funds in the bank.

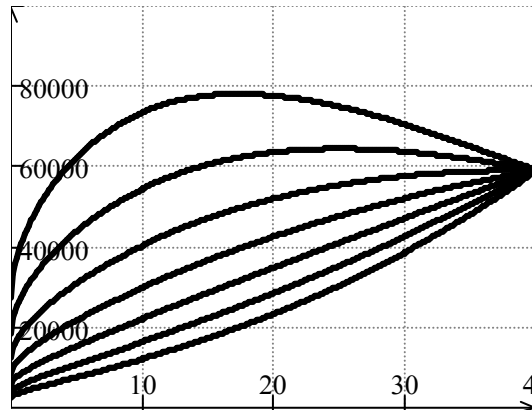
Although we could use the second-derivative test to verify that this gives a maximum in $A(t)$, the first-derivative test is probably easier, considering that two of the factors of $A'(t)$ must be strictly positive. We can note that the graph of $f(t) = \sqrt{t}$ is strictly increasing, so the graph of $1/\sqrt{t}$ must be strictly decreasing. Multiplying by the positive constant $0.5/2$ doesn't change that, nor does subtracting 0.06 . So $A'(t)$ must be decreasing everywhere, implying that at $t \approx 17.36$, the derivative must change from positive to negative—and therefore, that $A(t)$ achieves a maximum there.

5. Using $C = 2500$ and $r = 0.06$, plot the function $A(t)$ for several different values of k . What does a larger value of k imply about the value of the card over time? And now, what does a larger value of k imply about the best time to sell the card? Do these two facts seem consistent with one another?



In the graph above, $C = \$2500$ is held constant and $r = 0.06$ is held constant. k varies from 0 (lowest graph) to 0.6 (the graph that goes off the top of the image) in increments of 0.1. We see from the graphs that the lower k is—*i.e.*, the less rapidly the baseball card grows in value—the lower will be the amount of money the owner has for his retirement, regardless of when he sells it. That makes sense, since the slower the card appreciates the less it will always be worth. We also note that the local maxima for the graph occur at later times t when k is greater. That also makes sense—a card that is appreciating more rapidly is one you should want to hold onto longer.

6. Using $C = 2500$ and $k = 0.5$, plot the function $A(t)$ for several different values of r . What does a larger value of r imply about the best time to sell the card? Is that consistent with the meaning of r ?



In the graph above, $C = \$2500$ is held constant and $k = 0.5$ is held constant. r varies from 0 (lowest graph) to 0.06 (highest graph) in increments of 0.01. The most striking feature of the graph is that no matter what r is, the graph goes through the same point at time $t = 42$. But that makes sense, because $t = 42$ corresponds to the card owner not selling the card until he is ready to retire, in which case the interest it could have earned in the bank (but didn't) doesn't matter. We also note that for relatively low values of r , there is no advantage to selling the card before retirement, and for higher values of r , the optimal selling time is sooner the greater r is. That makes sense, because if your bank pays you lots of interest, then it is to your advantage to have money in the bank for a long time—hence, selling the card sooner.

7. Let t_{opt} be the optimal time to sell the baseball card—i.e., the time that will maximize $A(t)$. Use calculus to find t_{opt} in the general model (in terms of C , r , and k).

Recall that the model for $A(t)$ is $A(t) = (Ce^{k\sqrt{t}})e^{r(42-t)}$. This gives the general derivative

$$A'(t) = (Ce^{k\sqrt{t}})e^{r(42-t)}(-r) + \left(\frac{Ck^2 e^{k\sqrt{t}}}{2\sqrt{t}}\right)e^{r(42-t)} = (Ce^{k\sqrt{t}})e^{r(42-t)}\left(-r + \frac{k}{2\sqrt{t}}\right),$$

which leads to the general solution

$$t_{opt} = \frac{k^2}{4r^2}. \quad \text{To check, with } k = 0.5 \text{ and } r = 0.06, \text{ we have } t_{opt} = \frac{0.05^2}{4(0.06)^2} \approx 17.36.$$

(The first derivative test or the second derivative test can be used to verify that this yields a maximum; the approach would be the same as that described above, the only difference being the presence of parameters—since they're all known to be positive, the argument is the same.)

8. Are the properties of t_{opt} as it relates to k and r consistent with those you found in steps 5 and 6 above? Explain your answer.

The structure of $t_{opt} = \frac{k^2}{4r^2}$ is such that the best time to sell the card is later when k is greater (the card increases in value more rapidly) and is sooner when r is greater (the banks offer higher interest rates). Both of these are consistent with the conclusions drawn in the previous two steps.