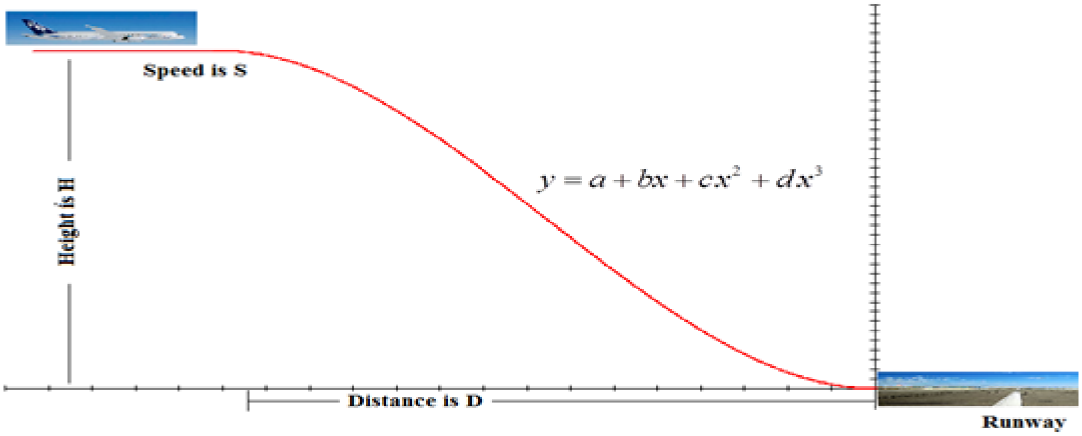
NCAAPMT Calculus Challenge 2011-2012

Challenge #6 SOLUTION

"Please return all seatbacks to their upright and locked positions. We will be landing parametrically."

An airliner is making its final landing approach and follows a cubic path, $y = a + bx + cx^2 + dx^3$, to smoothly transition from its horizontal flight path at altitude H to a horizontal landing path at ground level. Several conditions must be met to insure that the descent and landing are comfortable for the passengers.



a) The cruising altitude is H until the transition to the descent curve occurs at $(-D, H)$.

So, $\frac{dy}{dx} = 0$ at $(-D, H)$ and also at the landing site at (0, 0).

b) The horizontal airspeed, $\frac{dx}{dt}$, is a constant S mph throughout the descent (this is somewhat unrealistic, but is OK as an approximation).

c) At no time during the descent should the vertical component of acceleration exceed, in absolute value, some fixed constant k, $0 \le k \ll g$, where $g = 32 \ ft/\sec^2$ is the acceleration constant for gravity.

1. Find values of a, b, c, and d in the function $y = a + bx + cx^2 + dx^3$ which satisfy the conditions above in terms of D, S, and H.

If $y = a + bx + cx^2 + dx^3$ and (0, 0) is a point on the graph, then $a = 0$. Also, if $\frac{dy}{dx} = 0$ at $x = 0$, then $0 = b + 2cx + 3dx^2$ at $x = 0$, so $b = 0$. The form of the equation must be $y = cx^2 + dx^3$ with $x \in [-D, 0]$.

Now, $y(-D) = H$ and $y'(-D) = 0$, so $H = cD^2 - dD^3$ and $\frac{cD^2 - H}{D^3} = d$. Also $2c(-D) + 3dD^2 = 0$, giving $d = \frac{2c}{3D}$. So, $\frac{cD^2 - H}{D^3} = \frac{2c}{3D}$ and $c = \frac{3H}{D^2}$. Then $d = \frac{2c}{3D}$ gives $d = \frac{2H}{D^3}$.

Finally, we have $y=\left(\frac{3H}{D^2}\right)x^2+\left(\frac{2H}{D^3}\right)x^3$. If $x = 0$, then $y = 0$. If $x=-D$, then $y=H$, so that checks. Also, $y'(0)=0$ and $y'(-D)=0$, so we are good-to-go.

If we want to write this in terms of *S*, we know that $St=x$, so $y=\left(\frac{3H}{D^2}\right)(St)^2+\left(\frac{2H}{D^3}\right)(St)^3$ is height *y* as a function of *t*.

2. Determine the maximum vertical acceleration under the stated conditions in terms of the parameters in the problem (*D*, *H*, *S*, *k*).

Since height is given by $y=\left(\frac{3H}{D^2}\right)x^2+\left(\frac{2H}{D^3}\right)x^3$,
the vertical velocity is $\frac{dy}{dt}=\left(\frac{6H}{D^2}\right)x\frac{dx}{dt}+\left(\frac{6H}{D^3}\right)x^2\frac{dx}{dt}=\left(\frac{6HS}{D^2}\right)x+\left(\frac{6HS}{D^3}\right)x^2$.

The vertical acceleration is $\frac{d^2y}{dt^2}=\left(\frac{6HS}{D^2}\right)\frac{dx}{dt}+\left(\frac{12HS}{D^3}\right)x\frac{dx}{dt}=\left(\frac{6HS^2}{D^2}\right)+\left(\frac{12HS^2}{D^3}\right)x$.

So, $accel_y=\left(\frac{6HS^2}{D^2}\right)\left(1+\frac{2x}{D}\right)$. This is a linear equation with a positive slope. The maximum value is at the right endpoint of the domain $x\in[-D,0]$. So, the maximum acceleration is $k=\left(\frac{6HS^2}{D^2}\right)$ ft/sec^2.

It is always advisable to check units. *H* is in feet, as is *D*, so if *S* is in ft/sec, we have units of $\left(\frac{\text{ft}\times\frac{\text{ft}^2}{\text{sec}^2}}{\text{ft}^2}\right)=\text{ft/sec}^2$.

3. Suppose *S* = 600 mph, *H* = 35,000 feet, and $k = 0.4$ ft/sec^2. From what distance must the plane begin its descent?

First, 600 mph is $600\text{ mi/hr}\times\left(\frac{5280\text{ ft/mi}}{3600\text{ sec/hr}}\right)=880\text{ ft/sec}$. Now, $k=\left(\frac{6HS^2}{D^2}\right)$ means $D=\sqrt{\frac{6HS^2}{k}}$.

In this case, we have $D=\sqrt{\frac{6(35,000)(880)^2}{0.4}}\approx 637,621\text{ ft}$. This is about 121 miles from the airport, which seems about right from my cross country flying experience.

4. Suppose a small single engine plane is landing on a runway in the West Virginia mountains. It must fly over a mountain and immediately descend to the runway in the valley below. In this case, we have *S* = 150 mph, *H* = 12,000 feet, and *D* = 10 miles. What is the value of *k* for this landing?

In this case $k = \left(\frac{6HS^2}{D^2} \right)$ with $S = 220$ ft/sec . So, $k = \left(\frac{6(12,000)(220)^2}{52800^2} \right) \approx 1.25$ ft/sec^2 . You will “feel” this descent more than the slower descent in described above.

Reference: Barshinger, Richard, *How Not to Land at Lake Tahoe!*, **The American Mathematical Monthly**, May, 1992, Volume 99, Number 5, pp. 453-455.