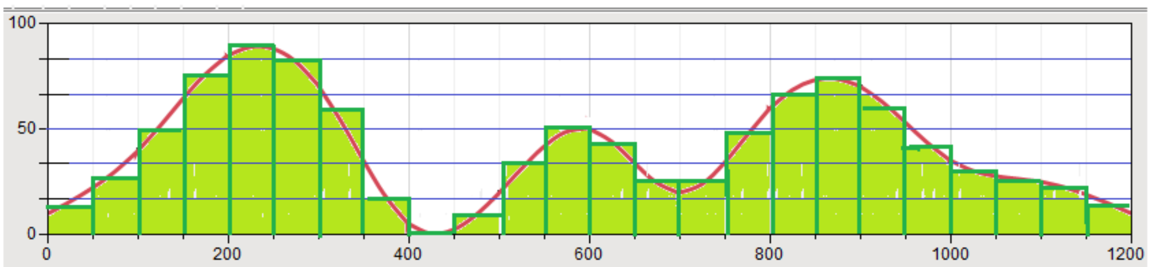
NCAAPMT Calculus Challenge 2011-2012

Challenge #7 SOLUTION

1) Use the photo to approximate the area of fabric you'll need. Be sure to explain your method and state whether you think your approximation is too large or too small.

We can approximate the amount of fabric needed by using a midpoint approximation for the area under the curve. Other approximations (rectangles at endpoints or trapezoids) will also work.

When I printed out the graph on paper, it was 6 cm tall (0-100), so I used 1 centimeter as 50/3 meters vertically.



The estimates of the heights at the midpoints (measured in 50/3 meters) of each 50 meter interval are:

Interval	1	2	3	4	5	6	7	8	9	10	11	12
Height (*gu*)	0.8	2.0	3.0	3.5	4.2	3.7	3.0	1.0	0	0.7	2.0	2.8
Height (*m*)	13.3	33.3	50	58.3	70	61.7	50	16.7	0	11.7	33.3	46.7
Interval	13	14	15	16	17	18	19	20	21	22	23	24
Height (*gu*)	2.6	1.5	1.5	2.6	4.2	4.5	3.5	2.5	1.8	1.6	1.2	1.0
Height (*m*)	43.3	25	25	43.3	70	75	58.3	41.7	30	26.7	20	16.7

The total for the estimated area in gu is $55.2\ gu^2$. Of course, many other estimates are possible.

This corresponds to $55.2\ gu^2\left(\frac{50}{3}\frac{m(height)}{gu}\right)50\frac{m(width)}{gu} = 46{,}000\ m^2$.

Exhibition Hall

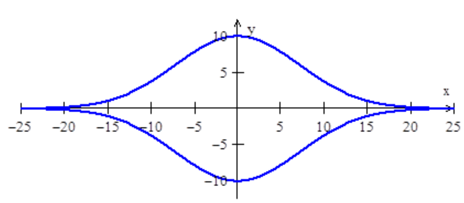
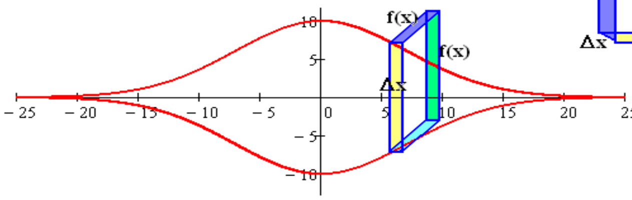
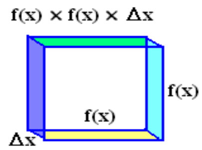


Figure 2: $f(x) = 10e^{-\frac{x^2}{100}}$ and $g(x) = -10e^{-\frac{x^2}{100}}$

The boundaries of the building are modeled by the functions *f* and *g* shown above on the domain $[-25, 25]$. You would like to build an exhibition hall whose base is the region defined above and whose vertical cross-sections are all perfect squares.

2) Compute the total interior volume of the proposed exhibition hall.





Each vertical element of the volume has a base that is Δx meters wide by $2 \cdot f(x_i)$ meters long. The element is a square parellelpiped with a height that is also $2 \cdot f(x_i)$ meters. So, the volume of the i^{th} volume element, V_i, is $V_i = \left(2 \cdot f(x_i)\right)^2 \Delta x$.

We add these up across the x-axis from −25 to 25. So, $V \approx \sum_{i=1}^{n} V_i = 4\sum_{i=1}^{n} f(x_i)^2 \Delta x$. Taking the limit as $\Delta x \to 0$, we have $V = \lim_{\Delta x \to 0} 4\sum_{i=1}^{n} f(x_i)^2 \Delta x = 4\int_{-25}^{25} f(x_i)^2\, dx$. This volume is 5013.254 square meters.

$$f(x) := 10 \cdot e^{\frac{-x^2}{100}} \qquad \int_{-25}^{25} (2f(x))^2\, dx = 5013.254$$

3) Determine the average height of the building.

The average height of the building is the ratio of its volume to the area of the base. The area of the base is $2\int_{-25}^{25} f(x)\,dx = 354.35$ square meters. So, $\bar{H} = \dfrac{4\int_{-25}^{25} f(x)^2\,dx}{2\int_{-25}^{25} f(x)\,dx} = 14.148$ meters.

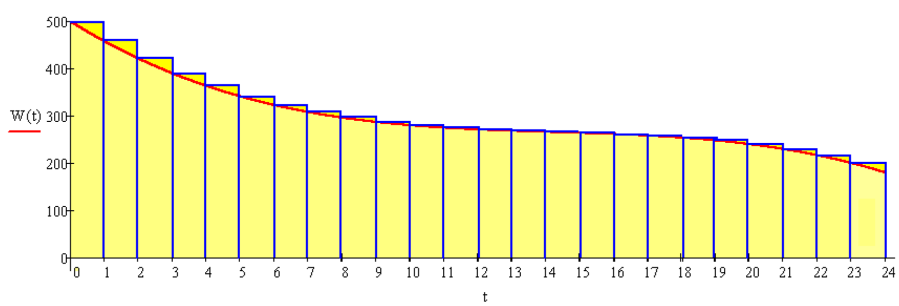
Building the exhibition hall

You determine that it will take 24 months to build it. However, the number of workers required is not constant over that time period. You estimate that if $t = 0$ is the time when construction begins, then the number of workers required at month t (where $0 \le t \le 24$) is approximately given by the function $w(t) = -0.07t^3 + 3t^2 - 45t + 500$. It costs \$3000 to hire a worker for a month. (We say that \$3000 is the cost of a "worker-month".)

4) Estimate how much it will cost to employ all the workers that it will take to build the exhibition hall.

The estimate of the cost can be obtained in a variety of ways, depending upon the assumptions made about hiring.

If we assume workers are hired at the beginning of the month for the month, then the total cost for the project is estimated to be the area of the rectangles in the diagram at right. This area is $\sum_{k=0}^{23} W(k) = 7219.68$ (we are using the left-endpoint for the number of workers needed for the month).

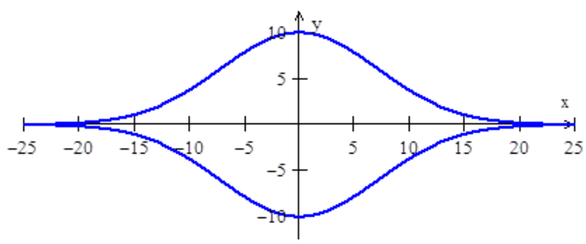


The units are worker-months. At \$3000 per worker-month, the cost is estimated at $\$3000\cdot\sum_{k=0}^{23} W(k) = \$21,659,040$. The area under curve can be used to approximate the area of the rectangles, so we could use $\$3000\int_0^{24} W(t)\,dt = \$21,173,760$ as the cost for workers on the project. Notice that this estimate is significantly lower than the previous estimate.

If we hire workers for a week, then we can use the estimate $\$3000\cdot\sum_{k=0}^{24\cdot 4-1} W\left(\frac{k}{4}\right) = \$21,294,000$ or by the day (assuming a 6-day work work for the project) and use $\$3000\cdot\sum_{k=0}^{24\cdot 24-1} W\left(\frac{k}{4\cdot 6}\right) = \$21,193,750$.

Grand Opening!

The exhibition hall is about to open! For the grand opening, you've decided to string lanterns completely around the perimeter of the building. Since vertical cross-sections of the building are squares, the footprint of the building on the ground has a similar profile to the vertical profile; it's as wide as it is tall at every point:



$$f(x) = 10e^{-\frac{x^2}{100}} \text{ and } g(x) = -10e^{-\frac{x^2}{100}}$$

Lanterns will be placed at every 50 cm (0.5 meters) around the entire perimeter of the building.

5) Determine how many lanterns will be needed.

To determine the number of lights needed, we must first compute the length of the boundary. We know that the arc length for function f from $x = a$ to $x = b$ is given by $L = \int_a^b \sqrt{1 + f'(x)^2}\,dx$. We have $f(x) = 10e^{-\frac{x^2}{100}}$, so

$f'(x) = 10e^{-\frac{x^2}{100}}\cdot\left(\frac{-2x}{100}\right) = -\frac{xe^{-\frac{x^2}{100}}}{5}$. That means our integrand is $\sqrt{1 + \frac{x^2e^{-\frac{x^2}{50}}}{25}}$, and with the symmetries about both axes, we can evaluate $4\int_0^{25}\sqrt{1 + \frac{x^2e^{-\frac{x^2}{50}}}{25}}\,dx = 111.235$ meters around.

To place a lantern every half-meter, they will need 222 lamps and have one gap that is a bit too wide.

Thanks to Floyd Bullard, Instructor of Mathematics, NC School of Science and Mathematics, for this bi-week's problem.