NCAAPMT Calculus Challenge 2011-2012

Challenge #8 **Due: February 15, 2012**

Note change of date

When using mathematics to model real-world situation, you often have to work from data. Given a set of data, what function models the data and what are the best parameter values given you observations. Sometimes, as with linear models, this is quite straight-forward. Just enter the data in a calculator or spreadsheet, and do a linear regression. At other times, it is quite difficult, particularly when there have been shifts in the data that make the standard fitting programs unusable (for example, most calculators will find a model for $y = ae^{bx}$ but not for $y = ae^{bx} + c$. How would you fit a model of the second form if you had no information about the value of *c*?

In this investigation, we will start with a simple problem where our technique using difference quotients and differential equations, while useful, is not necessary. Then we will move on to more involved models.

Part I: Ball Thrown from top of building

To find a model for this data, we first compute the slopes of the lines between data points. We will use the symmetric difference for these models. As a consequence, there will be two fewer data points for the slopes than for the original data.

First use data analysis to find a differential equation of the form $\frac{dh}{dt} = at + b$. Then solve the differential equation to find *h* as a function of *t*.

Time (sec)	0.2	0.6	0.9	1.4	2.1	2.7	3.2	3.6	4.0
Height (feet)	96.1	99.4	100.8	101.2	97.4	90.2	82.5	73.3	63.6
$\frac{\Delta Height}{\Delta Time}$	X	$\frac{100.8-96.1}{0.9-0.2}=6.71$	$\frac{101.2-99.4}{1.4-0.6}=2.25$					$\frac{63.6-82.5}{4.0-3.2}=-23.63$	X

1. Calculate the remaining difference quotients to approximate the derivative of height with respect to time. We now have data representing the derivative of Height with respect to time with ordered pairs (0.6, 6.71), (0.9, 2.25), …, (3.6, -23.63).

   Since for most functions modeling real-world phenomenon, the derivative is almost always more simple than the original function, we can more easily fit a model to the derivative, then solve the resulting differential equation to find our model.

2. Find a linear equation $\frac{\Delta H}{\Delta t} = Ax + B$ to model the difference quotient data.

3. If $\frac{dH}{dt} = Ax + B$, with the values of *A* and *B* from above, solve the differential equation to find ***height*** as a function of ***time***. Don't forget the constant of integration. Use one of the original data points to find the value of that constant.

How well does this function model the original data? Notice your model is an approximation since there is always error in the data. The value for the force of gravity, *g*, is off by just a bit.

Part II: Cooling Data

Elapsed time (min)	15	22	24	27	32	37	40	50	64
Temp (degrees F)	130	122	119	116	112	108	106	100	94
$\frac{\Delta Temp}{\Delta Time}$	X								X

A warm cup of water is cooling towards room temperature. We know that this should be modeled with an exponential function of the form $Temp = A \cdot e^{-Bt} + C$, with *C* representing the ambient room temperature. Unfortunately, we don't know what the room temperature was when the data was collected.

Calculate symmetric difference quotients to approximate the derivative of Temperature with respect to time.

We can model this data using two methods.

4. First, we know that constants disappear when we take derivatives. Since $Temp = A \cdot e^{-Bt} + C$, we know that $\frac{dTemp}{dt} = -BA \cdot e^{-Bt}$. Use your calculator to find an exponential model to the difference quotients. *Be careful, the range is now negative*, and your calculator takes logs when it does an exponential fit, so you need to fit the model to ordered pairs from $-\frac{dTemp}{dt} = BA \cdot e^{-Bt}$. Use the difference quotients and fit a model to (22, +1.22), …(50, +0.5) . Also, note that your calculator finds a model of the form $y = a\left(b^x\right)$, so you will need to rewrite using $b^x = e^{\ln(b)x}$ and solve. Find the values of A and *B* from the regression model, and use a point to solve for the missing value of *C*.

5. A more sophisticated approach for the cooling data is to recognize that exponential functions have the property that the rate of change is proportional to the value of the dependent variable. That is, they satisfy the differential equation $\frac{dy}{dt} = ky$. In our case, since $Temp = A \cdot e^{-Bt} + C$ and $\frac{dTemp}{dt} = -BA \cdot e^{-Bt}$, we have $\frac{dTemp}{dt} = k \cdot \left(Temp - C\right)$ for some constants *k* and *C*. This means the graph of our difference quotients graph against *Temp* rather than *Time*, should yield a line. Use the slope and intercept of the line and an initial condition from a data point should allow us to find the parameter values, *A*, *B*, and *C*, for our model.

6. Comment on how well the two methods do by comparing the functions created to the original data.

Part III: Population Data

The data below present the US population from 1790 to 1910 according to the US census.

Years since 1790 (t)	0	10	20	30	40	50	60	70	80	90	100	110	120
U.S. pop (P) in millions	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4	38.6	50.2	63	76.2	92.2
$\left(\frac{\Delta P}{\Delta t}\right)$	X												X
$\frac{\left(\frac{\Delta P}{\Delta t}\right)}{P}$	X												X

A logistic growth model of the form $P = \frac{M}{1 + Ae^{Mkt}}$, where M is the maximum sustainable population, is represented by the differential equation $\frac{dP}{dt} = kP(M - P)$. The value of A comes from the initial condition (0, 3.9). We can rewrite the differential equation $\frac{dP}{dt} = kP(M - P)$ as $\left(\frac{1}{P}\right)\frac{dP}{dt} = kM - kP$. If a logistic model is appropriate, then the graph of $\left(\frac{1}{P}\right)\frac{dP}{dt}$ plotted against the population P will be linear. The ordered pairs $\left(P, \frac{\left(\frac{\Delta P}{\Delta t}\right)}{P}\right)$ should be a line with a slope of k, and an intercept of kM.

7. Use the data above and the symmetric difference quotients to find values of k and M for the model $P = \frac{M}{1 + Ae^{Mkt}}$. Comment on how well the model fits the given data.