Stirling's Formula

When \( n \) is large, an efficient way to approximate \( n! \) is needed. Stirling’s formula approximates \( n! \) with \( \sqrt{2\pi n} e^{-n} n^n \). This formula is very useful in Mathematical Statistics to prove many of the limit theorems in that subject. In this challenge, we will derive Stirling’s approximation formula.

1. First, show that \( n! \approx e \cdot n^n e^{-n} \) gives a rough approximation for \( n! \).

Given \( n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \),

a) Show that \( \ln(n!) \approx \int_1^n \ln(x) \, dx \).

Consider \( n! = n(n-1)(n-2)\cdots(3)(2)(1) \). By taking logs we have

\[
\ln(n!) = \ln(n) + \ln(n-1) + \ln(n-2) + \cdots + \ln(3) + \ln(2) + \ln(1).
\]

The sum on the right side of the equation can be viewed geometrically as the area of the \( n-1 \) rectangles under the graph of \( y = \ln(x) \). This area can be approximated by the area under the curve as shown below.

[Graph showing rectangles approximating \( \ln(x) \) near \( x = 1 \) to \( x = 6 \).]

\[
\ln(n!) = \ln(n) + \ln(n-1) + \ln(n-2) + \cdots + \ln(3) + \ln(2) + \ln(1) \approx \int_1^n \ln(x) \, dx
\]

So \( \ln(n!) \approx \int_1^n \ln(x) \, dx \).

b) Evaluate \( \int_1^n \ln(x) \, dx \) and show that \( n! \approx e \cdot n^n e^{-n} \).

The integral \( \int_1^n \ln(x) \, dx \) is another standard integration by parts integral. The value is

\[
\int_1^n \ln(x) \, dx = n \ln(n) - n + 1.
\]

Since \( \ln(n!) \approx \int_1^n \ln(x) \, dx \), we have \( \ln(n!) \approx n \ln(n) - n + 1 \).
Solving for \( n! \), we find \( (n!) \approx e^{n \ln(n) - n} = e^1 e^{-n} e^{\ln(n)} = e \cdot e^{-n} n^n \).

2. For a better approximation, we need a more refined definition for the factorial. The factorial function is defined recursively by
   \[
   F(n) = \begin{cases} 
   1 & \text{if } n = 0 \\
   n \cdot F(n-1) & \text{if } n > 0 \quad \text{and } n \in \mathbb{Z}^+. 
   \end{cases}
   \]
   a) Given \( G(n) = \int_0^\infty x^n e^{-x} \, dx \), show that \( G(n) = n \cdot G(n-1) \).

   We are given \( G(n) = \int_0^\infty x^n e^{-x} \, dx \). To evaluate the integral, use integration by parts. So
   \[
   u = x^n \quad \text{and} \quad dv = e^{-x} \, dx \\
u = x^n \text{ and } v = -e^{-x}
   \]
   and
   \[
   \int_0^\infty x^n e^{-x} \, dx = \lim_{k \to \infty} \left[ (-x^n e^{-x}) \right]_0^k + n \int_0^\infty x^{n-1} e^{-x} \, dx = 0 + n \int_0^\infty x^{n-1} e^{-x} \, dx
   \]
   This means that \( G(n) = n \cdot G(n-1) \). That’s the basic definition of the factorial. If we can also show that \( G(0) = 1 \), then the function defined by our integral will indeed be the factorial function we know and love.

b) Show further that \( G(0) = 1 \), and so \( n! = \int_0^\infty x^n e^{-x} \, dx \).

   The integral \( G(0) = \int_0^\infty x^0 e^{-x} \, dx \). This is a standard improper integral.
   \[
   \int_0^\infty e^{-x} \, dx = \lim_{k \to \infty} \int_0^k e^{-x} \, dx = \lim_{k \to \infty} (-e^{-k} + 1) = 1. \quad \text{The function } F(n) = \int_0^\infty x^n e^{-x} \, dx \text{ is the factorial function for all positive integers } n. \text{ (This also gives another explanation for why } 0! = 1.)
   \]
   Use your calculator to evaluate \( \int_0^\infty x^n e^{-x} \, dx \) for \( n = 3, 4, 5, 6, \ldots \) (you can use 100 for the upper limit of integration). This factorial function is a variation of a function known as the gamma function \( \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx \) which has important applications in statistics.

3. Now that we know \( n! = \int_0^\infty x^n e^{-x} \, dx \), we can derive Stirling’s approximation formula.
a) Show that \( n! = \int_0^\infty e^{n \ln(x) - x} \, dx \). 

This just requires rewriting \( x^n e^{-x} \) as \( e^{n \ln(x) - x} = e^{n \ln(x) - x} \).

b) Show that the value of \( x \) for which \( h(x) = n \ln(x) - x \) has its maximum is \( x = n \).

If \( h(x) = n \ln(x) - x \), then \( h'(x) = \frac{n}{x} - 1 \). This means that \( h'(x) = 0 \) at \( x = n \). Since \( h''(x) = \frac{-n}{x^2} \), we know that \( h''(n) < 0 \), so the maximum value of \( h \) is obtained when \( x = n \).

c) Approximate \( h(x) = n \ln(x) - x \) by the quadratic function \( Q(x) = a + b(x - n) + c(x - n)^2 \).

Find values of \( a \), \( b \), and \( c \) which satisfy these three conditions,

i) \( Q(n) = h(n) \),
ii) \( Q'(n) = h'(n) \), and
iii) \( Q''(n) = h''(n) \).

Since \( Q(x) = a + b(x - n) + c(x - n)^2 \), \( Q'(x) = b + 2c(x - n) \), and \( Q''(x) = 2c \), we have

\[
Q(n) = a, \quad Q'(n) = b, \quad \text{and} \quad Q''(n) = 2c.
\]

Also, \( h(x) = n \ln(x) - x \), so \( h(n) = n \ln(n) - n \), \( h'(n) = 0 \) (from part b) above), and \( h''(n) = -\frac{1}{n} \). Satifying the three conditions above means that

i) \( a = n \ln(n) - n \),
ii) \( b = 0 \), and
iii) \( 2c = -\frac{1}{n} \), so \( c = -\frac{1}{2n} \).

The quadratic approximation we seek is \( Q(x) = n \ln(n) - n - \frac{(x - n)^2}{2n} \).

Our approximation is now \( n! = \int_0^\infty e^{h(x)} \, dx \approx \int_0^\infty e^{n \ln(n) - n - \frac{(x - n)^2}{2n}} \, dx \).

d) If \( n! \approx \int_0^\infty e^{Q(x)} \, dx \), show that \( n! \approx n^n e^{-\frac{n}{2}} \int_0^\infty e^{-\frac{(x - n)^2}{2n}} \, dx \).

We are integrating with respect to \( x \), so \( n \) is a constant in the integral, so we can pull it outside the integration.

\[
\int_0^\infty e^{n \ln(n) - n - \frac{(x - n)^2}{2n}} \, dx = \int_0^\infty e^{n \ln(n) - n - \frac{(x - n)^2}{2n}} \, dx = \int_0^\infty e^{n \ln(n) - n - \frac{(x - n)^2}{2n}} \, dx = e^{-\frac{n^2}{2n}} \int_0^\infty e^{-\frac{(x - n)^2}{2n}} \, dx.
\]
The rest of the problem is evaluating the remaining improper integral, which is sketched out for you.

First, make the substitution \( y = x - n \), so the integral is \( \int_{-n}^{\infty} e^{-\frac{y^2}{2}} dy \). Now let \( u = \frac{y}{\sqrt{n}} \) (we could have just used one substitution, but it’s more difficult to see how the choice was made). The integral is now

\[
\sqrt{n} \int_{-\sqrt{n}}^{\infty} e^{-\frac{u^2}{2}} du .
\]

Then, for any reasonably large value of \( n \), this integral has the same value as \( \sqrt{n} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du \), since \( e^{-\frac{u^2}{2}} \approx 0 \) for \( u > 5 \). This final integral is a standard of multivariable calculus and has the value

\[
\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi} .
\]

Putting it all together, we have Sterling’s formula,

\[
n! \approx e^{-n^n} \sqrt{2\pi n} .
\]

**Optional Material:** A quick description of the integration \( \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi} .
\)

Let \( A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \). Due to the symmetry of the function, this area is twice that of \( \int_{0}^{\infty} e^{-\frac{x^2}{2}} dx \), so \( \int_{0}^{\infty} e^{-\frac{x^2}{2}} dx = \frac{A}{2} \). Then, \( \left( \int_{0}^{\infty} e^{-\frac{x^2}{2}} dx \right) \cdot \left( \int_{0}^{\infty} e^{-\frac{y^2}{2}} dy \right) = \frac{A^2}{4} \), since \( x \) and \( y \) are just dummy variables in the integration. We now rewrite this product as a double integral

\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{(x^2 + y^2)}{2}} dy \ dx = \frac{A^2}{4} .
\]

(Rewriting the product of the two integrals as the double integral of the product of the integrands is a step that needs more justification than we give here, although the result is easily believed. It is straightforward to show that \( \int_{0}^{M} f(x) \ dx \int_{0}^{M} g(y) \ dy = \int_{0}^{M} \int_{0}^{M} f(x)g(y) \ dy \ dx \) for finite limits of integration, but the infinite limits create a significant challenge that will not be taken up.)

The double integral can be evaluated using polar coordinates, with \( x^2 + y^2 = r^2 \) and \( dy \ dx \) replaced with \( rd\ r \ d\theta \).

\[
\int_{0}^{\infty} \int_{0}^{\pi/2} e^{-\frac{r^2}{2}} r \ dr \ d\theta = \int_{0}^{\infty} \int_{0}^{\pi/2} e^{-\frac{r^2}{2}} r \ dr \ d\theta .
\]

To evaluate the polar form requires a \( u \)-substitution in an improper integral. Performing the integration with respect to \( r \), we have

\[
\int_{0}^{\pi/2} \int_{0}^{\infty} e^{-\frac{r^2}{2}} r \ dr \ d\theta = \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-u^2} \ du \ d\theta = \int_{0}^{\pi/2} d\theta = \frac{\pi}{2} .
\]
Now we know that $\frac{A^2}{4} = \frac{\pi}{2}$, and so $A = \sqrt{2\pi}$. 