NCAAPMT Calculus Challenge 2011-2012

Challenge #2 SOLUTION

Who Needs Calculus for Derivatives?

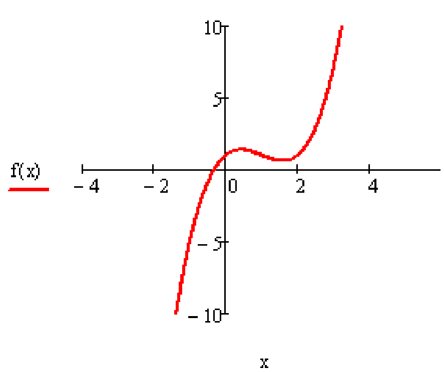
One of the first things you learn in Calculus is how to find derivatives and to write the equation of a line tangent to a curve. In this challenge, we will try to convince you that you didn't need to learn calculus to do either.

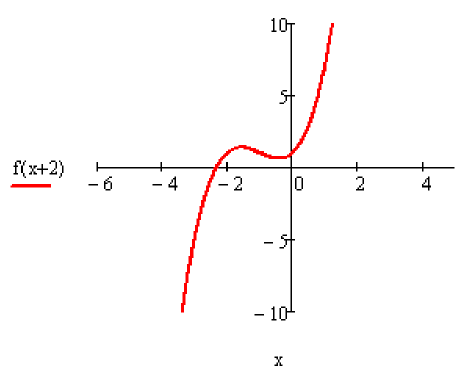
1) First, use the methods of calculus to find the equation of the line tangent to $f(x)=ax^3+bx^2+cx+d$ at $x=0$. Now, compare the equation of the tangent line to that of the polynomial. Is the result a surprise? Use calculus to prove that the line tangent to any polynomial at $x=0$ is always the linear component of the polynomial.

We know that $f(0)=d$ and $f'(x)=3ax^2+2bx+c$, so $f'(0)=c$. Since the tangent line has an equation of $L(x)=f(a)+f'(a)(x-a)$, this gives tangent of $L(x)=d+cx$. We notice that this is just the linear and constant terms of the original polynomial. If we think about this it is not a surprise. Close to $x = 0$, powers of x are very small, so the polynomial will be dominated by the linear and constant terms.

In general, we have $P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ and $P(0)=a_0$ with $P'(0)=a_1$, so the equation of the tangent line is $L(x)=a_0+a_1x$, as expected.

2) Suppose we want to find the line tangent to $f(x)=5x^3-3x^2+x-5$ at $x=2$. In 1) above, we saw that it is easy to find the tangent at $x=0$. A horizontal shift doesn't change the shape of a curve, so would it work to shift the curve over so that the tangent we want is now at $x=0$, use the method in 1), then shift back into its natural position? If this method works, explain why and write a general proof. If it doesn't always work, explain why not.





For $f(x)=5x^3-3x^2+x-5$ at $x=2$, we have $f(2)=25$ and $f'(x)=15x^2-6x+1$, so $f'(2)=49$, so our tangent line is $L(x)=25+49(x-2)$.

Now, shifting $f(x)=5x^3-3x^2+x-5$ to the left 2 units, we have $S(x)=f(x+2)=5(x+2)^3-3(x+2)^2+(x+2)-5=5x^3+27x^2+49x+25$. The tangent line at zero is $L^*(x)=25+49x$. Moving this two units to the right gives $L^*(x-2)=L(x)=25+49(x-2)$. This shift to zero, write down the equation, and shift back will always work.

Proof: Let $P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ so $P(b)=a_0+a_1b+a_2b^2+\cdots+a_nb^n=\sum_{k=0}^{n}a_kb^k$. Also $P'(x)=a_1+2a_2x+3a_3x^2\cdots+na_nx^{n-1}$ and $P'(b)=a_1+2a_2b+3a_3b^2\cdots+na_nb^{n-1}=\sum_{k=0}^{n}k\cdot a_kb^{k-1}$. So, our tangent line at $x = b$ has the equation $L(x)=\left(\sum_{k=0}^{n}a_kb^k\right)+\left(\sum_{k=0}^{n}k\cdot a_kb^{k-1}\right)(x-b)$.

We need to show that this the equation we generate using our shift method.

If $P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$, then $P(x+b)=a_0+a_1(x+b)+a_2(x+b)^2+\cdots+a_n(x+b)^n$. Fortunately, we only need to know the constant and linear terms of this polynomial.

The constant is $a_0+a_1(b)+a_2(b)^2+\cdots+a_n(b)^n=\sum_{k=0}^{n}a_kb^k$. The linear term is more difficult. Expand the binomials just a bit.

$$P(x+b)=a_0+a_1(x+b)+a_2\left(x^2+2bx+b^2\right)^2+a_3\left(x^3+\cdots 3xb^2+b^3\right)$$
$$+a_4\left(x^4+\cdots+4xb^3+b^4\right)+\cdots+a_n\left(x^n+\cdots nxb^{n-1}+b^n\right)$$

The linear term is $+a_1x+a_22bx^2+a_33xb^2+a_44xb^3+\cdots+a_nnxb^{n-1}=x\sum_{k=0}^{n}k\cdot a_kb^{k-1}$.

So, the line at zero is $L^*(x)=\left(\sum_{k=0}^{n}a_kb^k\right)+x\left(\sum_{k=0}^{n}k\cdot a_kb^{k-1}\right)$. Now we move it back, so that $L(x)=L^*(x-b)=\left(\sum_{k=0}^{n}a_kb^k\right)+\left(\sum_{k=0}^{n}k\cdot a_kb^{k-1}\right)(x-b)$.

There is another way to find derivatives using the concept of the slope of a tangent line without resorting to the limit definition of derivative.

3) Let $f(x) = ax^2$. Suppose we want to know the derivative of f. We know that the derivative at any value of x will allow us to write the equation of a line tangent to the curve. This means that there is a slope M and a line $y = Mx + B$ that is tangent to $f(x) = ax^2$. We want to know the value of M for which the equation $ax^2 = Mx + B$ has only one solution, since the curves are tangent and intersect at only one point. Use this idea to show that for $f(x) = ax^2$, $M = 2ax$ and consequently, that $\frac{d}{dx}\left(ax^2\right) = 2ax$.

Since $ax^2 = Mx + B$ must have only one value of x as a solution if the line is tangent to the curve, we require $ax^2 - Mx - B = 0$ to have a double root. So, $x = \dfrac{M \pm \sqrt{M^2 + 4aB}}{2a}$ with the discriminant $M^2 + 4aB = 0$ (so $M^2 = -4aB$).

Now, with a single solution, we have $x = \dfrac{M}{2a}$, so $M = 2ax$.

4) Try the method of single point of intersection developed in 3) on $f(x) = ax^2 + bx + c$.

This method works for the general quadratic as well. If $ax^2 + bx + c = Mx + B$ has only one solution, then $ax^2 + (b - M)x + (c - B) = 0$ so $x = \dfrac{(M - b) \pm \sqrt{(b - M)^2 - 4a(c - B)}}{2a}$ with $(b - M)^2 = 4a(c - B)$. So, $x = \dfrac{(M - b)}{2a}$ and $M = 2ax + b$.

5) Now let $g(x) = \dfrac{a}{x}$. Use the method describe in 3) to show that if $\dfrac{a}{x} = Mx + B$ has only one solution, then $M = \dfrac{-a}{x^2}$ (this problem is more challenging than the last).

If $\frac{a}{x} = Mx + B$ has a single solution, then $Mx^2 + Bx - a = 0$ has a double root. So,

$x = \frac{-B \pm \sqrt{B^2 + 4aM}}{2M}$ with $B^2 = -4aM$ (if $a > 0$, $M < 0$). If $x = \frac{-B}{2M}$, then $2Mx = -B$.

We can use $B^2 = -4aM$ here or, since $\frac{a}{x} = Mx + B$, $B = \frac{a}{x} - Mx$.

If $2Mx = -B$ and we use $B^2 = -4aM$, then $4M^2x^2 = -4aM$ so $M = \frac{-a}{x^2}$.

If $2Mx = -B$ and we use $B = \frac{a}{x} - Mx$, then $2Mx = -\frac{a}{x} + Mx$ and $Mx = -\frac{a}{x}$ or $M = \frac{-a}{x^2}$.

6) Use the method of single point of intersection on $h(x) = a\sqrt{x}$.

Consider $Mx + B = a\sqrt{x}$ with only one solution. So, $(Mx + B)^2 = a^2 x$ and we want a double root to $M^2x^2 + (2MB - a^2)x + B^2 = 0$.

So, $x = \frac{(a^2 - 2MB) \pm \sqrt{(2MB - a^2)^2 - 4M^2B^2}}{2M^2}$ where $(2MB - a^2)^2 - 4M^2B^2 = 0$.

Then $x = \frac{(a^2 - 2MB)}{2M^2}$ and $2M^2x = a^2 - 2MB$. We need to substitute for *a* and *B*. Since $(Mx + B)^2 = a^2x$, we have $\frac{(Mx + B)^2}{x} = a^2$, so $2M^2x = a^2 - 2MB$ can be rewritten as $2M^2x = M^2x + \frac{B^2}{x}$. This is equivalent to $M^2x^2 = B^2$. Since $B = a\sqrt{x} - Mx$ from the original equation, we have $M^2x^2 = a^2x - 2aMx\sqrt{x} + M^2x^2$. Adding out terms, we have

$a^2x = 2aMx\sqrt{x}$. Solving for *M*, we find, $\frac{a}{2\sqrt{x}} = M$.

I’m sure there is a nicer way to do this, but that’s what I came up with.

7) For what kinds of functions would the single solution to $f(x) = Mx + B$ approach fail?

Any function for which an algebraic solution to $f(x) = Mx + B$ is possible where exactly one solution is allowed. This rules out transcendental functions like $e^x = Mx + B$, $\sin(x) = Mx + B$, and $\ln(x) = Mx + B$ and general polynomials (but we already know how to do polynomial derivatives without calculus).