

NCAAPMT Calculus Challenge 2011-2012

Challenge #3

Due: October 26, 2011

A Model of Traffic Flow

Everyone has at some time been on a multi-lane highway and encountered road construction that required the traffic to occupy only one lane each way. Naturally, the Department of Transportation would like to maximize the flow of traffic through this stretch of the highway. What speed limit should be set for such a stretch of road to ensure the greatest traffic flow while also maintaining safety?

When developing a mathematical model of a real-world situation, it is usually necessary to make some simplifying assumptions. In this model, we assume that all the cars are the same length L and that the cars follow each other at a distance d (see Figure 1). We know from experience that the faster we drive, the more distance we should leave between our car and the car in front of us. Therefore, we want our model to reflect the fact that the following distance d depends upon the assigned speed limit s .



Figure 1: Diagram of the flow of cars on the highway

Cars are flowing uniformly down the road, each traveling at speed s and leaving a distance of d between them and the car in front. One simple model of traffic would be the equation

$$F = \frac{s}{L + d}. \quad (2)$$

We know that the following distance d , depends on the speed of the cars.

Three different “rules of thumb” are commonly used to determine a safe following distance. You might want to check your state’s Driver’s Manual to see what is recommended. The faster you are going the greater the distance must be between you and the car in front of you to give you time and distance to safely stop if the car in front of you stops suddenly.

We first note that while highway speed limits are typically given in miles per hour in the U.S., car lengths are not generally estimated in miles. Therefore, it will be convenient to build into our model a unit conversion of s from miles per hour to feet per second. Since there are 5280 feet in a mile and 3600 seconds in an hour, the conversion factor is

speed (in feet per second) = $\frac{5280 \text{ feet/mile}}{3600 \text{ seconds/hour}} \times \text{speed (in miles per hour)}$, or $s_{fps} = 1.467 \cdot s_{mph}$.

Building this conversion into our model in equation (2) gives us the following form of the model:

$$F = \frac{1.467s_{mph}}{L + d}, \quad (3)$$

in which F is the rate of traffic flow *in cars per second*, L is the length of the cars *in feet*, d is the following distance *in feet*, and s_{mph} is the speed limit *in miles per hour*.

Rule 1: Follow 2 car lengths for every 10 mph.

It is easy to translate this rule into an equation:

$$d = \frac{s_{mph}}{10} \cdot 2L.$$

If we use this rule of thumb, our model of traffic flow becomes:

$$F = \frac{1.467s_{mph}}{L + \left(\frac{s_{mph}}{10} \cdot 2L\right)} = \frac{7.335s_{mph}}{L(5 + s_{mph})}. \quad (4)$$

We need to find the critical values for this model. For Model 1, we have $F = \frac{7.335s_{mph}}{L(5 + s_{mph})}$.

So $\frac{dF}{ds} = \frac{(5L + Ls)7.335 - 7.335s(L)}{(5L + Ls)^2} = \frac{7.335(5L + Ls - Ls)}{(5L + Ls)^2} = \frac{36.675}{(5 + s)^2}$. There are no realistic

critical values for this function. As s increases, the function approaches $F = \frac{7.335}{L}$ asymptotically. With this model, the maximum flow depends on L , and decreases as L increases.

Rule 2: Follow three seconds behind the car in front.

We saw in equation (3) that a car traveling at s_{mph} miles per hour is traveling at $1.467 \cdot s_{mph}$ feet per second. So in three seconds, such a car would travel a distance of $d = 3(1.467s_{mph})$ feet. If we use this rule of thumb, our model of traffic flow becomes:

$$F = \frac{1.467s_{mph}}{L + (3 \times 1.467s_{mph})} = \frac{1.467s_{mph}}{L + 4.401s_{mph}} \quad (5)$$

For Model 2, $F = \frac{1.467s_{mph}}{L + 4.401s_{mph}}$, so

$$\text{So } \frac{dF}{ds} = \frac{(L + 4.401s)(1.467) - (1.467s)(4.401)}{(L + 4.401s)^2} = \frac{1.467(L)}{(L + 4.401s)^2}.$$

For this model, as well, there are no realistic critical values. As s increases without bound, the function approaches $F = \frac{1.467}{4.401} = \frac{1}{3}$ asymptotically. With this model, the maximum flow is fixed at 0.333 cars/sec, or 20 cars/minute.

Rule 3: The data below describe the distance needed to stop at various speeds. We can use techniques of data analysis to determine appropriate stopping distances. Table 1 from a state's driver's handbook gives some approximate figures.

<u>Speed (mph)</u>	<u>Thinking Distance (ft)</u>	<u>Braking Distance (ft)</u>
20	20	20
30	30	45
40	40	80
50	50	125
60	60	180

Table1: Speed and stopping distances from driver's handbook

Notice that the stopping distance is broken into two parts, the distance in feet T that the car travels while the driver is reacting and putting his foot on the brake, and the distance in feet B that the car travels as the brake slows the car to a stop. This means that $d = T + B$. The traffic flow function can be written more specifically as

$$F = \frac{1.467s_{mph}}{L + T + B}, \tag{6}$$

where T and B are functions of s_{mph} to be determined from the data in Table 1. Clearly, $T = s_{mph}$. We can find function B by using data analysis. Since the data are non-linear and should pass through $(0, 0)$, a power model seems appropriate.

If we linearize the data with a log-log re-expression, we see that the re-expressed data is nicely linear with equation $\ln(B) = -2.996 + 2\ln(s_{mph})$ or $B = 0.05s_{mph}^2$. (See Figure 2.)

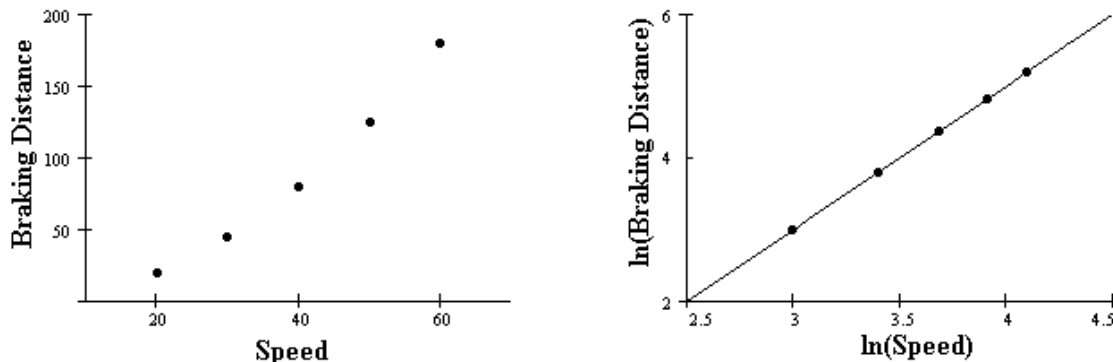


Figure 2: A log-log transformation linearizes the braking distance-speed data.

Students should recognize that the “data” in Table 1 are not really data at all, but values created from these functions. Actual data would not behave so nicely. (Some manuals do give actual data, but the functional models should be approximately equivalent.)

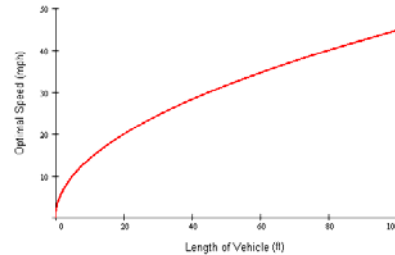
A model based on Rule 3 is
$$F = \frac{1.467s_{mph}}{L + s_{mph} + 0.05s_{mph}^2}.$$

Now,
$$\frac{dF}{ds} = \frac{(L + s + 0.05s^2)1.467 - 1.467s(1 + 0.1s)}{(L + s + 0.05s^2)^2} = \frac{1.467(L - 0.05s^2)}{(L + s + 0.05s^2)^2}.$$

There is one critical value when $s > 0$. This critical value is $s = \sqrt{\frac{L}{0.05}}$. Here, the maximum flow depends on L , and increases as L increases. Since $\frac{dF}{ds}$ changes from positive to negative as we cross $s = \sqrt{20L}$, we know we have a relative maximum values here.

2. The New Jersey Turnpike has lanes that are restricted for large truck travel only. Using Model 3, should the speed limit be set higher or lower for these lanes? Explain your answer.

Solution: If the vehicles are longer and the braking distances stay the same (of course they don't stay the same), then the optimal speed should increase, as shown in the graph of $s = \sqrt{20L}$ below.



3. Select the model that best describes traffic flow and which has a maximum flow. Since this model is dependent on the length of the vehicle, the maximum traffic flow will also be dependent on the length. Graph traffic flow vs. speed and identify the ordered pair that represents maximum traffic flow for each of the following vehicles:

A model based on Rule 3 is
$$F = \frac{1.467s_{mph}}{L + s_{mph} + 0.05s_{mph}^2}$$
 and we know that $s = \sqrt{20L}$ is the speed

in miles per hour which maximizes the flow of traffic in cars per minute.

- a) Mini Cooper length of 142.8 inches.
- b) Toyota Corolla length of 178.3 inches.
- c) Hummer H2 length of 189.9 inches.
- d) An 18-wheeler between 70 and 80 feet

Vehicle	Length	(s, F) for the maximum
Mini Cooper	142.8 inches	$(15.43 \frac{mi}{hr}, 0.577 \frac{coopers}{sec})$
Toyota Corolla	178.3 inches	$(17.25 \frac{mi}{hr}, 0.539 \frac{Corollas}{sec})$

Hummer H2	189.9 inches	$(17.79 \frac{mi}{hr}, 0.528 \frac{Hummers}{sec})$
Eighteen-wheeler	70 to 80 feet	$(38.73 \frac{mi}{hr}, 0.301 \frac{trucks}{sec})$

The length of 75 feet was used for the eighteen-wheeler.

3. The web site

http://www.thetruckersreport.com/truck_facts.shtml#Length%20of%20eighteen%20wheelers

gives the statement that “The length of time to stop an eighteen wheeler is 40% greater than that of an automobile.” Apply this information to find the ordered pair that represents the speed for maximum traffic flow for an eighteen-wheeler.

4. Our experience suggests that drivers do not leave as much space between cars as they should. Let p be the fraction of the required stopping distance that the drivers actually leave between cars. Assume $0.1 \leq p \leq 1$. Find the optimal speed and its corresponding traffic flow that maximizes traffic flow in terms of the parameters L and p . Which has a larger effect of the optimal speed, L or p ?

To model this situation, we can use the function $F = \frac{1.467s_{mph}}{L + p(T + B)}$ with $T + B = s + 0.05s^2$.

$$\text{So, } \frac{dF}{ds} = \frac{(L + p(s + 0.05s^2))1.467 - (1.467s)p(1 + 0.1s)}{(L + p(s + 0.05s^2))^2} = \frac{(1.467)(L - .05ps^2)}{(L + p(s + 0.05s^2))^2}.$$

The only reasonable critical value is located at $s = \sqrt{\frac{L}{0.05p}} = 4.472\sqrt{\frac{L}{p}}$. As p increases, the

optimal speed decreases, which is what we would expect to see. Also, as L increases (without affecting braking distance), the optimal speed increases.

Since $s \propto \sqrt{\frac{L}{p}}$, we see that $\frac{ds}{dp} \propto \frac{s}{p}$ while $\frac{ds}{dL} \propto \frac{s}{L}$. Since p is small while L is large, the

change in s due to small change in p will be more significant than the change in s due to small changes in L .

5. Some might argue that the drivers would never drive so close to the car in front of them that they would not have sufficient time to react. These drivers might (unconsciously, of course) leave all of the reaction distance but only a fraction of the braking distance. Adjust your model by multiplying braking distance by p , where $0.1 \leq p \leq 1$ to find the optimal speed and its corresponding traffic flow for the Corolla for $p = 0.1$, $p = 0.5$, and $p = 1$.

In this case,

$$F = \frac{1.467s_{mph}}{L + T + pB}.$$

How different from the previous solution is the optimal velocity if this model is used?

If $F = \frac{1.467s_{mph}}{L + T + pB}$, then $F = \frac{1.467s_{mph}}{L + s + 0.05ps^2}$.

$$\text{So, } \frac{dF}{ds} = \frac{(L + s + p(0.05s^2))1.467 - (1.467s)(1 + 0.1ps)}{(L + s + p(0.05s^2))^2} = \frac{(1.467)(L - .05ps^2)}{(L + s + 0.05ps^2)^2}.$$

The numerator in this derivative is the same as before, so the solution will be as well, with

$s = 4.472\sqrt{\frac{L}{p}}$. The flow rate of cars will differ, but the selected speed will be the same for both

models involving the parameter p . The Toyota Corolla has $L = 178.3$ inches or 14.86 feet.

p	Speed (mph)	Maximum Flow
0.1	55 mph	0.217 cars/sec or 13 cars/min
0.5	24 mph	0.108 cars/sec or 6.5 cars/min
0.8	19 mph	0.086 cars/sec or 5.2 cars/min
1.0	17 mph	0.078 cars/sec or 4.7 cars/min