Calculus Challenge #14
Calculus at the Battle of Trafalgar

The summer of 2005 marked the 200th anniversary of the British naval victory over a combined French and Spanish fleet in the waters off Cape Trafalgar. During the Napoleonic wars, naval warfare followed certain rules that seem rather formal to us today. The ships in each fleet lined up in a row sailing parallel to its opponent and fired as they sailed past each other (see Figure 1). This maneuver was repeated until one fleet was disabled or sunk. This is known as the directed fire model or conventional combat model.

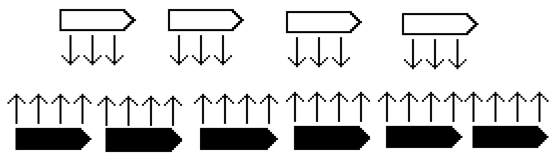

Figure 1: The White Fleet takes a beating

In such an engagement, the fleet with superior firepower will inevitably win. To model this battle, we begin with a system of differential equations that models the interaction of two fleets in combat. Suppose we have two opposing forces, fleet **A** with A_0 and fleet **B** with B_0 ships initially, and $A(t)$ and $B(t)$ ships t units of time after the battle is engaged. Given the style of combat at the time of Trafalgar, the losses for each fleet will be proportional to the effective firepower of the opposing fleet. That is,

$$\frac{dA}{dt} = -bB \text{ and } \frac{dB}{dt} = -aA,$$

where a and b are positive constants that measure the effectiveness of the ship's cannonry and personnel and A and B are both functions of time. These equations indicate that the rate at which one navy lost ships depended only on two things: the number of ships in the opposing fleet and the effectiveness of the opposition fire. It is assumed that the effectiveness does not change throughout the battle, so the rate at which a navy lost ships was proportional to the number of ships in the opposing fleet.

1) Assume that $a = b = k$ (the two fleets are equal in battle), and show that the total number of ships still fighting is decreasing exponentially by considering $\frac{d}{dt}(A + B)$.

2) Assume that $a = b = k$ and $A_0 > B_0$, and show that the difference in the size of the two fleets is increasing exponentially by considering $\frac{d}{dt}(A - B)$.

3) Use the results in 1) and 2) to solve for A and B as functions of time.

4) If $\frac{dA}{dt} = -bB$ and $\frac{dB}{dt} = -aA$, find $\frac{dA}{dB}$ and solve for *A* in terms of *B*. This equation will give you the expected number of ships remaining in fleet **A** when $B = 0$.

5) The commander of the British fleet was Admiral Nelson. In the now famous Battle of Trafalgar, he exhibited cunning military strategy. In one account of the battle, Nelson expected to have 27 ships in the British fleet (**B**) and predicted that the French/Spanish Armada (**A**) would have 34 ships. In planning his strategy, Nelson believed that the British fleet was better prepared (and better led) than the French/Spanish Armada. Suppose that $a = 0.75b$.

If Nelson's 27 ships fought a conventional battle against the 34 ships in the French/Spanish Armada with $a = 0.75b$, would he win? How many ships would remain in the winning fleet?

6) Instead of sailing parallel to the French/Spanish Armada, Nelson planned to sail through the middle of the fleet, cutting it in half and fighting two separate conventional battles. In one battle, he would have numerical superiority and consequently win that battle. In the other, he would have fewer ships and lose. But, with the ships that remained in the battle that he had won, would he be able to defeat the ships remaining in the French/Spanish fleet in the battle that they won? In a third and decisive battle, the British fleet would be victorious. It should be noted that Nelson assigned himself the task of leading the portion of his fleet that was expected to lose its battle.

Show, using the results from 4), how Nelson could arrange his 27 ships to defeat a larger fleet of 34 ships using a 3-battle plan as described above with $a = 0.75b$. According to our model, how many ships would be expected to survive the final battle?