Crossing a Street Using Calculus

If cars are moving along a one-lane street at a rate of 12 cars per minute, and it takes you 10 seconds to walk across the street, how long should you expect to wait before an interval between cars is large enough to allow you to cross the street? Problems like this belong to an area of mathematics known as queuing theory and use probabilistic models to describe daily events. Calculus plays an important role in queuing models, as we will see in this Challenge Problem. We will develop a general model using the intensity parameter λ and crossing time *T*, and then use the results to answer the question originally posed with $\lambda = 12$ and $T = 0.167$ (10 seconds). Notice that we need to be sure we are making all measurements in the same units, in this case minutes.

When we say that cars are moving along the road at a rate of 12 cars per minute, we don't really mean that a car arrives every 5 seconds. The measure 12 cars per minute is an average rate. We need a way to describe when we expect the individual cars to appear at our position on the side of the street. After watching traffic flow for many years, mathematicians have decided that the exponential distribution function $F(t) = \lambda e^{-\lambda t}$ is a good way to describe the interarrival time between cars. The arrival times can be found by summing the interarrival times. In free flowing traffic, the interarrival times are assumed to be independent. The function $F(t) = \lambda e^{-\lambda t}$, with $t > 0$, is a called a probability density (or distribution) function with intensity λ. A probability density function has two essential characteristics:

1) $F(t) \geq 0$ for all *t* in its domain.
2) The total area under the curve over the domain of the function is 1. This means $\int_0^\infty \lambda e^{-\lambda t} dt = 1$.

If we think of time 0 as the time a car has just passed your position, the probability that the next car arrives in the time interval $[t_0, t_1]$ is given by the area under the graph of *F* from t_0 to t_1.

That is, $P(\textit{Next car arriving between } t_0 \textit{ and } t_1) = \int_{t_0}^{t_1} \lambda e^{-\lambda t} dt$.

Figure 1 illustrates the problem setting. We will define gap G_k to be the gap between cars $k-1$ and *k*. Time $t = 0$ denotes the time when Car 0 has just passed your position. The cars are arriving at rate λ cars per minute with interarrival times distributed exponentially, and it takes T minutes for you to safely walk across the street. The cars are moving right to left in Figure 1.

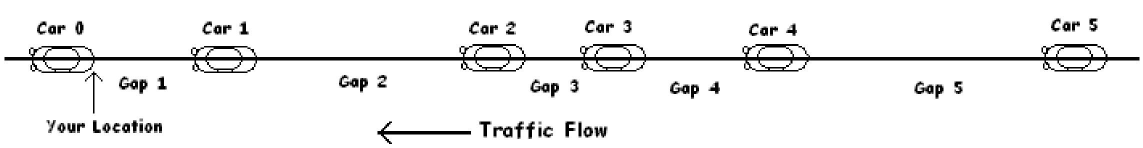

Figure 1: Diagram of cars and gaps

1. Find the probability that Car 1 arrives before time *T* (this is also the probability that Gap 1 is too short for you to cross in time).

2. Find the probability that Car 1 does not arrive before time *T* (this is also the probability that Gap 1 is long enough for you to cross in time).

3. Using the assumption that the interarrival times are independent, explain why the probability that Gap 4 is the first gap big enough for you to cross the street is given by $P(G_4) = \left(1 - e^{-\lambda T}\right)^3 \left(e^{-\lambda T}\right)$.

4. Using the notation $P(G_k)$ to represent the probability that the first gap with a length greater than *T* is Gap *k*, find $P(G_1)$, $P(G_2)$, $P(G_3)$, and the general probability $P(G_k)$. Also, compute $\sum_{i=1}^{\infty} P(G_k)$. Does this sum make sense?

The distribution of probabilities $P(G_k)$ for $k = 1, 2, 3, \ldots$ is known as a geometric distribution. It is a discrete probability distribution, since the domain is the positive integers. We want to know, on average, how many gaps we will need to look at before we find one long enough for us to cross the street. We need to find the expected value of *k*, denoted $\bar{k}$. The expected value of a discrete probability distribution is found by computing the infinite sum $\bar{k} = \sum_{i=1}^{\infty} k \cdot P(G_k)$. In this case, we need to compute the sum of $\sum_{i=1}^{\infty} k \cdot \overbrace{\left(e^{-\lambda T}\right)\left(1 - e^{-\lambda T}\right)^{k-1}}^{P(G_k)}$. Notice that this infinite series is not geometric, so we can't use our formula for the sum of an infinite geometric series. Fortunately, you have worked on Challenge Problem #4, so you have some techniques that allow you to find this sum by being clever and using your knowledge of derivatives.

5. Find the sum $\bar{k} = \sum_{i=k}^{\infty} k \cdot \left(e^{-\lambda T}\right)\left(1 - e^{-\lambda T}\right)^{k-1}$ to determine the average number of gaps before we can cross.

6. On average, how many cars will you watch go by on our street with $\lambda = 12$ and $T = 0.167$?
    a) How long would you expect this to take?
    b) How long would you expect to wait if you were on crutches and it took 30 seconds for you to cross the street?
    c) If you dashed across in 6 seconds?

7. In many places, if the expected wait is longer than 90 seconds, a crossing light may be installed to facilitate crossing. If $T = 0.167$, what must λ be for a crossing light to be needed?