Solutions to *Crossing the Street Using Calculus*

1. Find the probability that Car 1 arrives before time *T*.

The probability that Car 1 arrives before time *T* is $\int_0^T \lambda e^{-\lambda t}dt = 1 - e^{-\lambda T}$. This is the same for all cars. After a car has passed, the next car can be thought of as Car 1.

2. Find the probability that Car 1 does not arrive before time *T*.

The probability that Car 1 does not arrive before time *T* is $1-\left(1-e^{-\lambda T}\right)=e^{-\lambda T}$. This is also the same for all cars.

3. Using the assumption that the interarrival times are independent, explain why the probability that Gap 4 is the first gap big enough for you to cross the street is given by $P\left(G_4\right)=\left(1-e^{-\lambda T}\right)^3\left(e^{-\lambda T}\right)$.

If Gap 4 is the first gap big enough for you to cross, then Gaps 1, 2, and 3 must have been too small. If the interarrival times are independent, then the probability that Gap 1 and Gap 2 and Gap 3 are all too small is the product of the individual probabilities. So the probability that Gaps 1, 2, and 3 are too small and Gap 4 is large enough is the product of $P\left(G_4\right)=\left(1-e^{-\lambda T}\right)\cdot\left(1-e^{-\lambda T}\right)\cdot\left(1-e^{-\lambda T}\right)\cdot\left(e^{-\lambda T}\right)$. So $P\left(G_4\right)=\left(1-e^{-\lambda T}\right)^3\left(e^{-\lambda T}\right)$.

4. Using the notation $P\left(G_k\right)$ to represent the probability that the first gap greater than *T* is Gap *k*, find $P\left(G_1\right)$, $P\left(G_2\right)$, $P\left(G_3\right)$, and the general probability $P\left(G_k\right)$. Also, compute $\sum_{i=1}^{\infty} P\left(G_k\right)$. Does this sum make sense? By the same reasoning as in 4), we see that

$$P\left(G_1\right)=\left(e^{-\lambda T}\right)$$
$$P\left(G_2\right)=\left(1-e^{-\lambda T}\right)\left(e^{-\lambda T}\right)$$
$$P\left(G_3\right)=\left(1-e^{-\lambda T}\right)^2\left(e^{-\lambda T}\right)$$

and

$$P\left(G_k\right)=\left(1-e^{-\lambda T}\right)^{k-1}\left(e^{-\lambda T}\right)$$

The sum of the probabilities is $\sum_{i=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k-1}\left(e^{-\lambda T}\right)=\left(e^{-\lambda T}\right)\sum_{i=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k-1}$. Students should recognize this as an infinite geometric series with common ratio $\left(1-e^{-\lambda T}\right)$. So

$\left(1+\left(1-e^{-\lambda T}\right)+\left(1-e^{-\lambda T}\right)^2+\left(1-e^{-\lambda T}\right)^3+\cdots\right)=\frac{1}{1-\left(1-e^{-\lambda T}\right)}=\frac{1}{e^{-\lambda T}}$. Finally, the sum is $\left(e^{-\lambda T}\right)\sum_{i=1}^{\infty}\left(1-e^{-\lambda T}\right)^{k-1}=\frac{e^{-\lambda T}}{e^{-\lambda T}}=1$. We certainly expect the total probability to be 1, so this result is not a surprise.

5. To find the sum $\bar{k}=\sum_{k=1}^{\infty}k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$, first notice that the summand $k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$ looks very much like the derivative (with respect to *T*) of some function. Write the sum $\bar{k}=\sum_{k=1}^{\infty}k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$ a constant times the sum of a derivative. The summand $k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}$ is almost $\frac{d}{dT}\left(1-e^{-\lambda T}\right)^k$. Since $\frac{d}{dT}\left(1-e^{-\lambda T}\right)^k=k\left(1-e^{-\lambda T}\right)^{k-1}\left(-e^{-\lambda T}\right)\left(-\lambda\right)$, we see that the summand is $\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^k\right]$. So,

$$\bar{k}=\sum_{k=1}^{\infty}k\cdot\left(e^{-\lambda T}\right)\left(1-e^{-\lambda T}\right)^{k-1}=\left(\frac{1}{\lambda}\right)\sum_{k=1}^{\infty}\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^k\right].$$

We know that for finite sums, the sum of derivatives is the derivative of a sum. We won't prove that this infinite sum also obeys this rule, but it does. Rewrite the sum of derivatives as the derivative of a sum.

If the sum of the derivatives is equal to the derivative of the sum, then $\left(\frac{1}{\lambda}\right)\sum_{k=1}^{\infty}\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^k\right]=\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^k\right]$. Now the sum is of an infinite geometric series, so we know how to compute it.

. You should recognize that the sum is now an infinite geometric series. Find the sum, and then differentiate the result. Show that the average number of gaps before we can cross ($\bar{k}$) is $\bar{k}=e^{\lambda T}$. That means that you would expect to wait for $e^{\lambda T}-1$ cars to pass before you could cross.

Since $\left(\frac{1}{\lambda}\right)\sum_{k=1}^{\infty}\left[\frac{d}{dT}\left(1-e^{-\lambda T}\right)^k\right]=\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^k\right]$, we know the sum in this expression is $\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^k=\frac{1-e^{-\lambda T}}{1-\left(1-e^{-\lambda T}\right)}=\frac{1-e^{-\lambda T}}{e^{-\lambda T}}=e^{\lambda T}-1$. So $\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\sum_{k=1}^{\infty}\left(1-e^{-\lambda T}\right)^k\right]=\left(\frac{1}{\lambda}\right)\left[\frac{d}{dT}\left(e^{\lambda T}-1\right)\right]=\left(\frac{1}{\lambda}\right)\left(\lambda e^{\lambda T}\right)=e^{\lambda T}$, as required.

6. On average, how many cars will you watch go by on our street with $\lambda = 12$ and $T = 0.167$? How long would you expect this to take? How long would you expect to wait if you were on crutches and it took 30 seconds for you to cross the street? If you dashed across in 6 seconds?

In many places, if the expected wait is longer than 90 seconds, a crossing light may be installed to facilitate crossing. If $T = 0.167$, what must λ be for a crossing light to be needed?

7. If $\lambda = 12$ and $T = 0.167$, then we would expect to wait for $e^{12(0.167)} - 1 = e^2 - 1 \approx 6.4$ cars to pass. If cars arrive at a rate of one every 5 seconds, we would have to wait, on average, 32 seconds before we found a gap between cars large enough for us to cross. If you were on crutches and it took 30 seconds ($T = 0.5$), then you would expect to wait for $e^{12(0.5)} - 1 = e^6 - 1 \approx 402$ cars to pass. You should begin looking for an alternate route now. If you could dash across in 6 seconds, then you would expect to wait for only $e^{12(0.1)} - 1 = e^{1.2} - 1 \approx 2.3$ two or three cars before crossing.

If a crossing light is needed whenever the expected wait is longer than 90 seconds, for $T = 0.167$, the traffic intensity that would necessitate a crossing light can be found by solving $\left(e^{\lambda(0.167)} - 1\right)\left(\dfrac{1}{\lambda}\right) = 1.5$. This equation cannot be solve analytically, so we can use Newton's Method or another numerical technique to find $\lambda \approx 20.78$ as the solution.

References:

Mooney, Douglas, and Randall Swift, *A Course in Mathematical Modeling*, Mathematical Association of America, 1999.

Mesterton-Gibbons, Michael, *A Concrete Approach to Mathematical Modeling*, Addison-Wesley Publishing Company, Redwood City, CA, 1989.