Calculus Challenge #14
Calculus at the Battle of Trafalgar

The summer of 2005 marked the 200th anniversary of the British naval victory over a combined French and Spanish fleet in the waters off Cape Trafalgar. During the Napoleonic wars, naval warfare followed certain rules that seem rather formal to us today. The ships in each fleet lined up in a row sailing parallel to its opponent and fired as they sailed past each other (see Figure 1). This maneuver was repeated until one fleet was disabled or sunk. This is known as the directed fire model or conventional combat model.

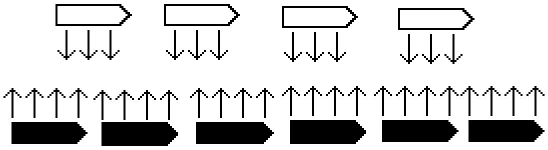

Figure 1: The White Fleet takes a beating

In such an engagement, the fleet with superior firepower will inevitably win. To model this battle, we begin with a system of differential equations that models the interaction of two fleets in combat. Suppose we have two opposing forces, fleet **A** with A_0 and fleet **B** with B_0 ships initially, and $A(t)$ and $B(t)$ ships t units of time after the battle is engaged. Given the style of combat at the time of Trafalgar, the losses for each fleet will be proportional to the effective firepower of the opposing fleet. That is,

$$\frac{dA}{dt} = -bB \text{ and } \frac{dB}{dt} = -aA,$$

where *a* and *b* are positive constants that measure the effectiveness of the ship's cannonry and personnel and *A* and *B* are both functions of time. These equations indicate that the rate at which one navy lost ships depended only on two things: the number of ships in the opposing fleet and the effectiveness of the opposition fire. It is assumed that the effectiveness does not change throughout the battle, so the rate at which a navy lost ships was proportional to the number of ships in the opposing fleet.

1) Assume that $a = b = k$ (the two fleets are equal in battle), and show that the total number of ships still fighting is decreasing exponentially by considering $\frac{d}{dt}\left(A + B\right)$.

2) Assume that $a = b = k$ and $A_0 > B_0$, and show that the difference in the size of the two fleets is increasing exponentially by considering $\frac{d}{dt}\left(A - B\right)$.

3) Use the results in 1) and 2) to solve for *A* and *B* as functions of time.

4) If $\frac{dA}{dt} = -bB$ and $\frac{dB}{dt} = -aA$, find $\frac{dA}{dB}$ and solve for A in terms of B. This equation will give you the expected number of ships remaining in fleet **A** when $B = 0$.

5) The commander of the British fleet was Admiral Nelson. In the now famous Battle of Trafalgar, he exhibited cunning military strategy. In one account of the battle, Nelson expected to have 27 ships in the British fleet (**B**) and predicted that the French/Spanish Armada (**A**) would have 34 ships. In planning his strategy, Nelson believed that the British fleet was better prepared (and better led) than the French/Spanish Armada. Suppose that $a = 0.75b$.

If Nelson's 27 ships fought a conventional battle against the 34 ships in the French/Spanish Armada with $a = 0.75b$, would he win? How many ships would remain in the winning fleet?

6) Instead of sailing parallel to the French/Spanish Armada, Nelson planned to sail through the middle of the fleet, cutting it in half and fighting two separate conventional battles. In one battle, he would have numerical superiority and consequently win that battle. In the other, he would have fewer ships and lose. But, with the ships that remained in the battle that he had won, would he be able to defeat the ships remaining in the French/Spanish fleet in the battle that they won? In a third and decisive battle, the British fleet would be victorious. It should be noted that Nelson assigned himself the task of leading the portion of his fleet that was expected to lose its battle.

Show, using the results from 4), how Nelson could arrange his 27 ships to defeat a larger fleet of 34 ships using a 3-battle plan as described above with $a = 0.75b$. According to our model, how many ships would be expected to survive the final battle?

Solutions:

1) Assume that $a = b = k$ (the two fleets are equal in battle), and show that the total number of ships still fighting is decreasing exponentially by considering $\frac{d}{dt}(A + B)$.

Solution: Assuming the combat effectiveness of the combatants is equal, when we add the two equations together, we get

$$\frac{dA}{dt} + \frac{dB}{dt} = -kB - kA\,.$$

Recall that the sum of derivatives is the derivative of a sum. So we actually have

$$\frac{d(A + B)}{dt} = -k(A + B).$$

This equation is a variation of the basic differential equation for exponential functions $\frac{dP}{dt} = -kP$, where $P = A + B$. The solution to $\frac{dP}{dt} = -kP$ is $P(t) = P_0 e^{-kt}$, and so the solution to our equation is

$$A(t) + B(t) = (A_0 + B_0)e^{-kt}\,.$$

This means that the **total** number of ships in the battle *decreases exponentially* over the time of the battle.

2) Assume that $a = b = k$ and $A_0 > B_0$, and show that the positive difference in the size of the two fleets is increasing exponentially by considering $\frac{d}{dt}(A - B)$.

Solution: In a similar manner, we can consider the difference in the two defining differential equations.

$$\frac{dA}{dt} - \frac{dB}{dt} = -kB + kA = k(A - B) \text{ is equivalent to } \frac{d(A - B)}{dt} = k(A - B)$$

and so,

$$A(t) - B(t) = (A_0 - B_0)e^{kt}\,.$$

The **difference** in the number of ships remaining in the winning fleet *A* and those remaining in the losing fleet *B* *increases exponentially* over the time of the battle.

3) Use the results in 1) and 2) to solve for *A* and *B* as functions of time.

Solution: We now have two equations in two unknowns, *A* and *B*. Solve for *A* by adding the two equations from 1) and 2) together to eliminate *B*, this yields

$$2A(t) = (A_0 + B_0)e^{-kt} + (A_0 - B_0)e^{kt}$$

and

$$A(t) = \frac{\left(A_0 + B_0\right)e^{-kt} + \left(A_0 - B_0\right)e^{kt}}{2}.$$

Solving for *B*, we find that

$$B(t) = \frac{\left(A_0 + B_0\right)e^{-kt} - \left(A_0 - B_0\right)e^{kt}}{2}.$$

4) If $a \neq b$, the techniques used in 1) - 3) cannot be used to determine the number of ships in each fleet as a function of time. If $\frac{dA}{dt} = -bB$ and $\frac{dB}{dt} = -aA$, find $\frac{d^2A}{dt^2}$, substitute, and solve for $A(t)$. There will be two solutions to the 2nd order differential equation. The general solution will be the sum of these solutions. Use this solution to find $B(t)$. Be sure to check to see that the solutions in 3) are special cases of this more general solution.

Solution: If $\frac{dA}{dt} = -bB$ and $\frac{dB}{dt} = -aA$, then $\frac{d^2A}{dt^2} = -b\frac{dB}{dt} = -b\left(-aA\right) = ab\,A$. This shows that the 2nd derivative of *A* is proportional to *A*. What function is this? Only the exponential function has this property. However, there are two solutions, since the exponent could be either positive or negative. Both $A_1(t) = C_1\left(e^{\sqrt{ab}\,t}\right)$ and $A_2(t) = C_2\left(e^{-\sqrt{ab}\,t}\right)$ satisfy the differential equation $\frac{d^2A}{dt^2} = ab\,A$. The general solution, then, is the sum of the two. So, $A(t) = C_1\left(e^{\sqrt{ab}\,t}\right) + C_2\left(e^{-\sqrt{ab}\,t}\right)$ is the general solution to the second order differential equation $\frac{d^2A}{dt^2} = ab\,A$.

Now, $\frac{dB}{dt} = -aA$. This means that $\frac{dB}{dt} = -a\left(C_1\left(e^{\sqrt{ab}\,t}\right) + C_2\left(e^{-\sqrt{ab}\,t}\right)\right)$. By integration, we find that $B(t) = -C_1\sqrt{\frac{a}{b}}\left(e^{\sqrt{ab}\,t}\right) + C_2\sqrt{\frac{a}{b}}\left(e^{-\sqrt{ab}\,t}\right)$.

Since we have $A(0) = A_0$ and $B(0) = B_0$, we can solve for $C_1 = \frac{A_0 - \sqrt{\frac{b}{a}}B_0}{2}$ and $C_2 = \frac{A_0 + \sqrt{\frac{b}{a}}B_0}{2}$. Notice that if $a = b = k$, we have the same solutions as in 3).

5) If $\frac{dA}{dt} = -bB$ and $\frac{dB}{dt} = -aA$, find $\frac{dA}{dB}$ and solve for A in terms of B. This equation will give you the expected number of ships remaining in fleet **A** when $B = 0$.

Solution: We know that $\frac{dA}{dt} = -bB$ and $\frac{dB}{dt} = -aA$. So, from the chain rule and inverse function rule, we have $\frac{dA}{dB} = \frac{dA}{dt} \cdot \frac{dt}{dB} = \frac{\frac{dA}{dt}}{\frac{dB}{dt}} = \frac{-bB}{-aA}$ and $\frac{dA}{dB} = \frac{bB}{aA}$. This is a separable equation, so $\int aA\,dA = \int bB\,dB$ and it follows that $aA^2 = bB^2 + c$.

The initial conditions give us $c = aA_0^{\,2} - bB_0^{\,2}$, so $A = \sqrt{\frac{b}{a}B^2 + A_0^{\,2} - \frac{b}{a}B_0^2}$. The end of the battle occurs when $B = 0$. Substituting into the equation above, we find that the number of ships in fleet A at the end of the battle is

$$A = \sqrt{A_0^2 - \frac{b}{a}B_0^2}\,.$$

If Fleet **B** wins, then the expected number of surviving vessels is $B = \sqrt{B_0^2 - \frac{a}{b}A_0^2}$. By using this equation, we can determine the expected number of ships remaining in the winning fleet at the end of the battle. For example, two equal forces ($a = b$) with $A_0 = 20$ and $B_0 = 15$ would result in a victory for **A** with 13 ships remaining after the battle. This surprisingly large number is a result of the two exponential functions above. The total number is decreasing exponentially, but the difference in the size of the fleets is increasing exponentially.

6) Suppose that $a = 0.75b$. If Nelson's 27 ships fought a conventional battle against the 34 ships in the French/Spanish Armada with $a = 0.75b$, would he win? How many ships would remain in the winning fleet?

Solution: If Nelson's 27 ships fought a conventional battle against the 34 ships in the French/Spanish Armada with $a = 0.75b$, then $B_0^2 - \frac{a}{b}A_0^2 = 27^2 - \frac{3}{4}\left(34^2\right) < 0$ and Nelson would lose. The expected number of ships remaining in the winning French/Spanish fleet would be $\sqrt{34^2 - \frac{4}{3}\left(27^2\right)} \approx 13.56$ or 14 ships. (We will consider a "fractional ship" as still capable of fighting if the fraction is greater than one-half.) This would not be a good day for Admiral Nelson and the British.

7) Show, using the results from 5), how Nelson could arrange his 27 ships to defeat a larger fleet of 34 ships using a 3-battle plan as described above with $a = 0.75b$. According to our model, how many ships would be expected to survive the final battle?

Solution: Show, using the results from 5) how Nelson could arrange his 27 ships to defeat a larger fleet of 34 ships using a 3-battle plan as described above. According to our model, how many ships would be expected to survive the final battle?

First Battle (*A Wins*)				**Second Battle (*B Wins*)**				**Final Battle**			**Winner**
British	Armada	Survive		British	Armada	Survive		British	Armada	Survive	
4	17	**16**		23	17	**18**		18	16	**11**	**B**
5	17	**16**		22	17	**16**		16	16	**8**	**B**
6	17	**16**		21	17	**15**		15	16	**6**	**B**
7	17	**15**		20	17	**14**		14	15	**5**	**B**
8	17	**14**		19	17	**12**		12	14	**2**	**A**
9	17	**13**		18	17	**10**		10	13	**6**	**A**

As can be seen from the table, there are many options for Nelson. However, he will want the first battle (the one he loses) to last as long as possible, so that the second battle will have the best chance of being concluded before the survivors of the first battle can rejoin the fight. Having 7 ships in the first battle and 20 in the second allows Nelson to defeat the larger fleet with an expected 5 ships surviving.

The actual battle had more ships and did not actually follow this outline, but Nelson’s original and creative strategy changed naval history.

References:

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