

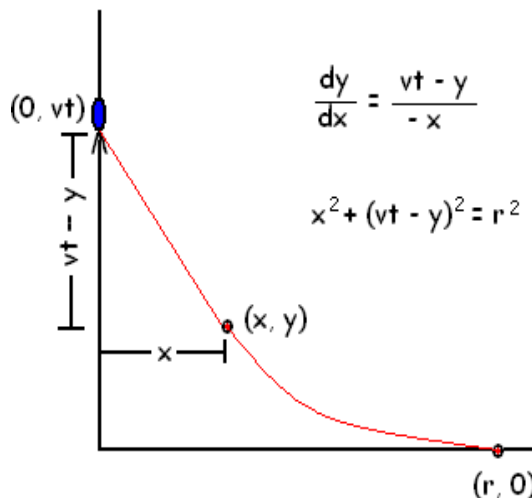
The Water-Skier Problem

Consider the situation in which a boat and water-skier are alongside a dock, connected by a tightly stretched rope. At time $t = 0$, the boat moves with a constant velocity v away from the dock in a direction perpendicular to the dock. Initially the skier is at $(r, 0)$ and the boat at $(0, 0)$. The rope has fixed length r and the boat is traveling up the y -axis with constant velocity v .

So, at time t , the boat is at $(0, vt)$ and the skier at $(x(t), y(t))$. We want to determine the path that the skier follows if

- i) the rope is always tangent to the skier's trajectory
- ii) the rope has a constant length r .

The figure at right describes the situation.



- a) Explain why, when the water-skier is at the point (x, y) , the boat is at $(0, vt)$, so $\frac{dy}{dx} = \frac{vt-y}{-x}$. (1 point)

From the description and the diagram, we see that the rope always points towards the boat and is always tangent to the curve. We can use this information to write a differential equation for the slope of this tangent line. When the water-skier is at the point (x, y) , the boat is at $(0, vt)$, so $\frac{dy}{dx} = \frac{vt-y}{0-x}$.

- b) This equation involves the variables $t, x,$ and y and the parameter v . Before we can solve the equations, we need to eliminate t . Explain why the equation can be rewritten as

$$\frac{dy}{dx} = \frac{-\sqrt{r^2 - x^2}}{x} \quad (1 \text{ pt})$$

This equation involves the variables $t, x,$ and y and the parameter v . Before we can solve the equations, we need to eliminate t . From the diagram we see that there is a right triangle formed with vertices $(x, y), (0, vt),$ and $(0, vt-y)$. So, $x^2 + (vt-y)^2 = r^2$.

So, we have two pieces of information, $\frac{dy}{dx} = \frac{vt-y}{0-x}$ and $x^2 + (vt-y)^2 = r^2$. Our present goal is to eliminate t , but we see that we have the expression $vt-y$ in both equations. Solving the second equation for $vt-y$ and substituting, we have $vt-y = \sqrt{r^2 - x^2}$ and so $\frac{dy}{dx} = \frac{-\sqrt{r^2 - x^2}}{x}$. By solving this equation we can find the path.

c) To solve the differential equation, use u -substitution on the integral $\int \frac{-\sqrt{r^2 - x^2}}{x} dx$. (1 pt)

The integral $\int \frac{-\sqrt{r^2 - x^2}}{x} dx$ can be evaluated using a u -substitution. Let $u = \sqrt{r^2 - x^2}$. The work is easier if we use $u^2 = r^2 - x^2$, then we don't have to deal with the square rooty part of the problem. If $u^2 = r^2 - x^2$, then $x = \sqrt{r^2 - u^2}$ and $2u du = -2x dx$, so $dx = \frac{-2u du}{2x} = \frac{-u du}{\sqrt{r^2 - u^2}}$.

$$\text{So, we have } \int \frac{-\sqrt{r^2 - x^2}}{x} dx = \int \left(\frac{-u}{\sqrt{r^2 - u^2}} \right) \left(\frac{u du}{\sqrt{r^2 - u^2}} \right) = \int \frac{u^2}{r^2 - u^2} du.$$

Can we integrate $\int \frac{u^2}{r^2 - u^2} du$?

d) After the u -substitution, you still have some work to do to complete the integration. (1 pt)

This integral, $\int \frac{u^2}{r^2 - u^2} du$, has the degree of the numerator the same as the degree of the denominator.

We need to divide to reduce the order of the integrand. By division we find that

$$\int \frac{u^2}{r^2 - u^2} du = \int -1 + \frac{r^2}{r^2 - u^2} du.$$

This second integral requires integration by partial fractions with $\frac{A}{r - u} + \frac{B}{r + u}$ and eventually

$$\int -1 + \left(\frac{r}{2} \right) \frac{1}{r - u} + \left(\frac{r}{2} \right) \frac{1}{r + u} du. \text{ Finally, an integral we can evaluate.}$$

Solving, we find that

$$\int -1 + \left(\frac{r}{2} \right) \frac{1}{r - u} + \left(\frac{r}{2} \right) \frac{1}{r + u} du = -u - \left(\frac{r}{2} \right) \ln(r - u) + \left(\frac{r}{2} \right) \ln(r + u) + C = -u + \left(\frac{r}{2} \right) \ln \left(\frac{r + u}{r - u} \right) + C.$$

Rewriting in terms of x and y , we have $y = -\sqrt{r^2 - x^2} + \left(\frac{r}{2} \right) \ln \left(\frac{r + \sqrt{r^2 - x^2}}{r - \sqrt{r^2 - x^2}} \right) + C$ as the path taken by the water-skier.

e) Use the initial condition to find the solution to the path of the water-skier. (1 pt)

At the start, we know that $x = r$ and $y = 0$, so $C = 0$. Finally, we have a solution to the water-skier

problem: if $\frac{dy}{dx} = \frac{-\sqrt{r^2 - x^2}}{x}$, then $y = -\sqrt{r^2 - x^2} + \left(\frac{r}{2}\right) \ln\left(\frac{r + \sqrt{r^2 - x^2}}{r - \sqrt{r^2 - x^2}}\right)$ is a solution to the equation.

The path of the water-skier is graphed below along with the “solutions” found by the TI-89 and Maple. Notice that there are no values of x and y in the first quadrant if you use the “solution” from either the TI-89 or Maple.

Solving, we find that:

$$y = -\sqrt{r^2 - x^2} + \left(\frac{r}{2}\right) \ln\left(\frac{r + \sqrt{r^2 - x^2}}{r - \sqrt{r^2 - x^2}}\right).$$

Result from TI-89:

$$TI(x) = r \ln|x| - r \ln(-r + \sqrt{r^2 - x^2}) - \sqrt{r^2 - x^2}$$

Result from Maple:

$$h(x) = -\sqrt{r^2 - x^2} + \left(\frac{r^2}{\sqrt{r^2}}\right) \text{Tan}^{-1}\left(\frac{r^2}{\sqrt{r^2} \cdot r^2 - x^2}\right)$$

