Calculus Challenge #13 — Due April 15, 2008

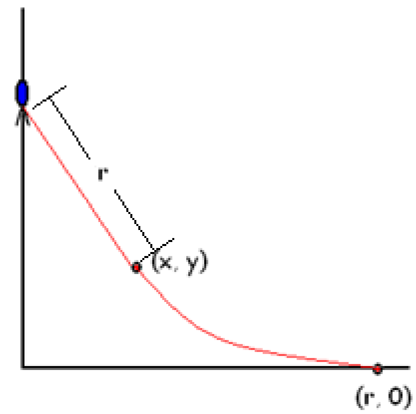
The Water-Skier Problem

Consider the situation in which a boat and water-skier are alongside a dock, connected by a tightly stretched rope. At time $t = 0$, the boat moves with a constant velocity v away from the dock in a direction perpendicular to the dock. Initially the skier is at $(r, 0)$ and the boat at $(0, 0)$. The rope has fixed length r and the boat is traveling up the y-axis with constant velocity v.

So, at time t, the boat is at $(0, vt)$ and the skier at $(x(t), y(t))$. We want to determine the path that the skier follows if

i) the rope is always tangent to the skier's trajectory
ii) the rope has a constant length r.

The figure at right describes the situation.



a) Explain why, when the water-skier is at the point (x, y), the boat is at $(0, vt)$, so $\frac{dy}{dx} = \frac{vt - y}{-x}$. (1 point)

b) This equation involves the variables t, x, and y and the parameter v. Before we can solve the equations, we need to eliminate t. Explain why the equation can be rewritten as

$$\frac{dy}{dx} = \frac{-\sqrt{r^2 - x^2}}{x}. \quad \text{(1 pt)}$$

c) To solve the differential equation, use u-substitution on the integral $\int \frac{-\sqrt{r^2 - x^2}}{x} dx$. (1 pt)

d) After the u-substitution, you still have some work to do to complete the integration. (1 pt)

e) Use the initial condition to find the solution to the path of the water-skier. (1 pt)

Food for thought: If you have access to a TI-89 or other similar calculator, have it do the integral $\int \frac{-\sqrt{r^2 - x^2}}{x} dx$. How does its solution compare to your correct solution in e)?