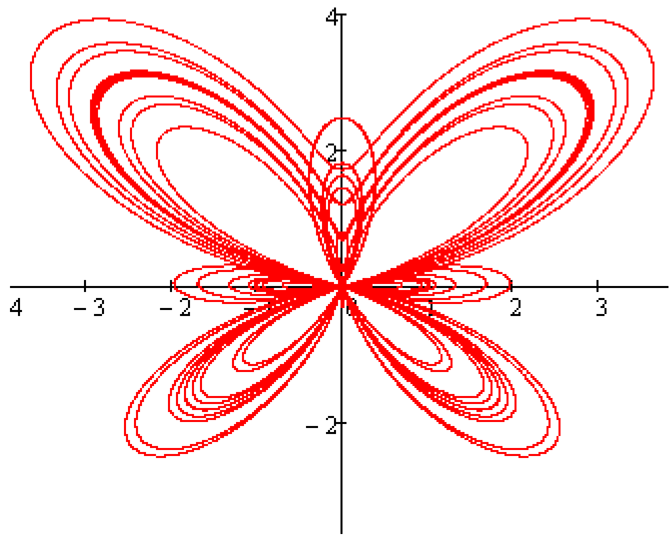
NCAAPMT Calculus Challenge Problem #10 SOLUTION

Use computer software or your calculator to sketch the graph of the curve defined by the parametric equations

$$x = \sin(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$

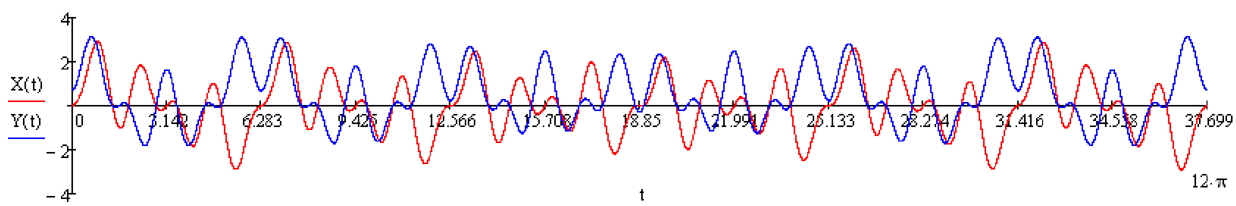
$$y = \cos(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$

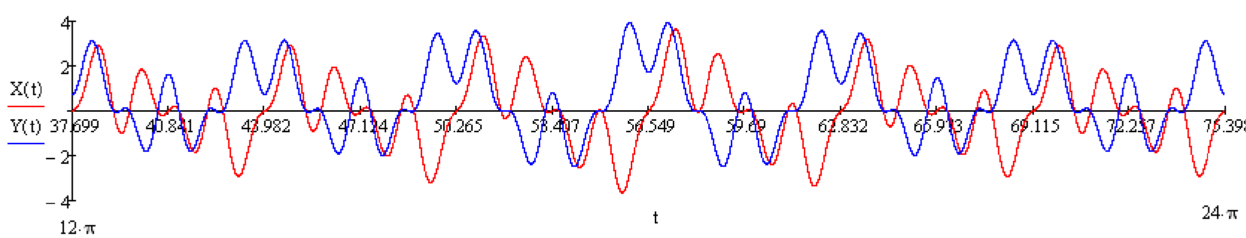
This curve is known as a "butterfly curve".



a) The curve is periodic. The graph will simply continue to travel over itself repeatedly. What is the period of this curve? (1 pt)

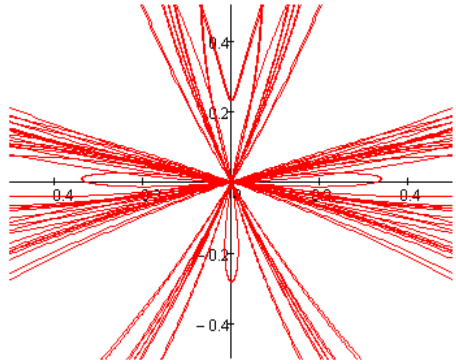
The periods for each trigonometric component differ. Usually, we need the lowest common multiple for all, which is the period for $\sin\left(\frac{t}{12}\right)$ which is 24π. Looking at the graph of the individual functions from 0 to 24π is informative. As you can see on the separate graphs of $x(t)$ and $y(t)$, each repeats after 24π.





b) How many times does the curve pass through the origin in one complete period? (1 pt)

The origin is seen in the figure at right.



The graph above also answers this question. We can see that the graphs of $x(t)$ and $y(t)$ are simultaneously zero 72 times. The curve makes 6 passes through the origin every 2π .

c) Write the equation of the line tangent to the curve at $t = 135\pi$. (2 pts)

OK. So here is the messy calculus you have come to expect. We need a point and a slope. Since the period is 24π , we see that $t = 135\pi$ is the same as $t = 15\pi$. The point is just $\left(x(15\pi), y(15\pi)\right)$. Which is $\left(0, \frac{e\left(\sqrt{2}-16\right)+8}{8e}\right)$ by direct computation. The slope requires two derivatives and a computation.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

$$\frac{dx}{dt} = \sin(t)\left(e^{\cos(t)}(-\sin t) + 8\sin(4t) - \frac{5}{12}\sin^4\left(\frac{t}{12}\right)\cos\left(\frac{t}{12}\right)\right) + \cos(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right).$$

and

$$\frac{dy}{dt} = \cos(t)\left(e^{\cos(t)}(-\sin t) + 8\sin(4t) - \frac{5}{12}\sin^4\left(\frac{t}{12}\right)\cos\left(\frac{t}{12}\right)\right) - \sin(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$

Evaluate at $t = 15\pi$ to find that $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{5e\sqrt{2}}{12e\sqrt{2} - 192e + 8}$. (there are lots of other equivalent expressions for this)

Finally, the equation of the line is $y = \left(\frac{5e\sqrt{2}}{12e\sqrt{2} - 192e + 8}\right)x + \frac{e\left(\sqrt{2}-16\right)+8}{8e}$.

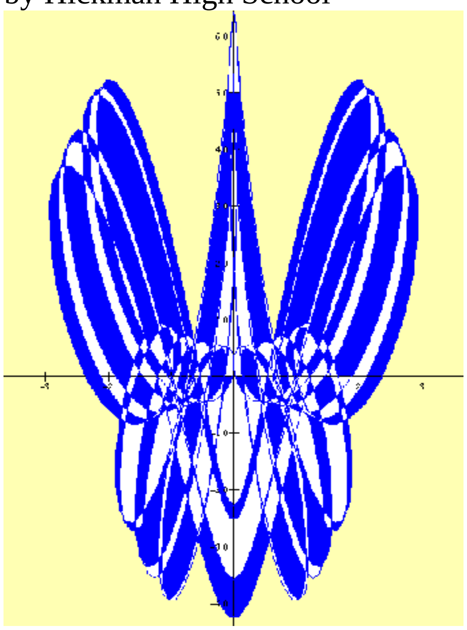
d) Modify the equations by changing the power on the $\sin\left(\frac{t}{12}\right)$ term, the coefficient in $\cos(4t)$, and the coefficient of the $\sin\left(\frac{t}{12}\right)$ term. Which has the greatest effect on the shape of the curve? Find the variation you like the most, color it and submit it with your solution. (1 pt)

The coefficient of $\cos(4t)$ affects the number of wings and the symmetry of the butterfly. The other coefficients affect the size of the wings and other appendages.

Two very nice renditions are shown below.

$$x = \sin(t)\left(e^{\cos(t)} - \cos(4t) - \sin^2\left(\frac{t}{12}\right)\right)$$
$$y = \cos(t)\left(e^{\cos(t)} - 5\cos(4t) - 2\sin^3\left(\frac{t}{12}\right)\right)$$

by Hickman High School



$$x = \sin(t)\left(e^{\cos(t)} - 2\cos(.5t) - \sin^5\left(\frac{t}{12}\right)\right)$$
$$y = \cos(t)\left(e^{\cos(t)} - 2\cos(t) - \sin^5\left(\frac{t}{12}\right)\right)$$

by The Westminster School