NCAAPMT Calculus Challenge Problem #10

SOLUTION

For each of the functions below, use the information about the function to determine its equation.

1) Let g be a differentiable function , defined for all real numbers x, with the following properties:

(i) $g'(x) = ax^2 + bx$ (ii) $g'(1) = 6$ and $g''(1) = 18$

(iii) $\int_1^2 g(x)\,dx = 18$ Find $g(x)$. (2 pts)

We know that $g'(1) = 6$ and $g''(1) = 18$ so $a + b = 6$ and $2a + b = 18$. This gives $a = 12$ and $b = -6$. Since $g'(x) = ax^2 + bx$, we know that $g(x) = a\frac{x^3}{3} + b\frac{x^2}{2} + c = 4x^3 - 3x^2 + c$ and if $\int_1^2 g(x)\,dx = 18$ then $c = 10$.

So, our function is $g(x) = 4x^3 - 3x^2 + 10$

2) Let h be a differentiable function , defined for all real numbers $x \geq 0$ such that $h(0) = 5$ and $h(3) = -1$. Suppose that for any number $b > 0$ the average value of $h(x)$ on the interval $0 \leq x \leq b$ is $\frac{h(0) + h(b)}{2}$.

a) Find $\int_0^3 h(x)\,dx$. (1 pt)

We know the average value is $\frac{\int_0^3 h(x)\,dx}{3-0}$, so this is $\frac{h(0)+h(3)}{2} = 2$. Then $\frac{\int_0^3 h(x)\,dx}{3-0} = 2$ gives $\int_0^3 h(x)\,dx = 6$.

b) Prove that $h'(x) = \frac{h(x)-5}{x}$. (1 pt)

From a), we see that $\frac{\int_0^x h(t)\,dt}{x-0} = \frac{h(0)+h(x)}{2}$. So, $\int_0^x h(t)\,dt = \frac{(h(0)+h(x))x}{2} = \frac{5x+xh(x)}{2}$.

Differentiating both sides with respect to *x* (using the 2nd Fundamental Theorem of Calculus, we have

$$h(x) = \frac{5+xh'(x)+h(x)}{2}, \text{ so } h(x) = 5+xh'(x). \text{ Solving for } h'(x) \text{ gives } h'(x) = \frac{h(x)-5}{x}.$$

c) Use parts a) and b) to find $h(x)$. (1 pt)

Since $h'(x) = \frac{h(x)-5}{x}$, we have $\frac{dy}{dx} = \frac{y-5}{x}$. This differential equation is separable. So, $\frac{1}{y-5}\frac{dy}{dx} = \frac{1}{x}$. Integrating both sides with respect to *x*, we have $\int \frac{1}{y-5}\frac{dy}{dx}dx = \int \frac{dx}{x} + c$ and $\ln|y-5| = \ln|x| + c$. Exponentiating both sides, we have $|y-5| = C|x|$. We are given the point $(3,-1)$, so $|-6| = C|3|$, so $C = 2$.

For the initial condition we have $y-5<0$ and $x>0$, so $|y-5| = 2|x|$ becomes $5-y = 2x$.

The equation is $y = 5-2x$.