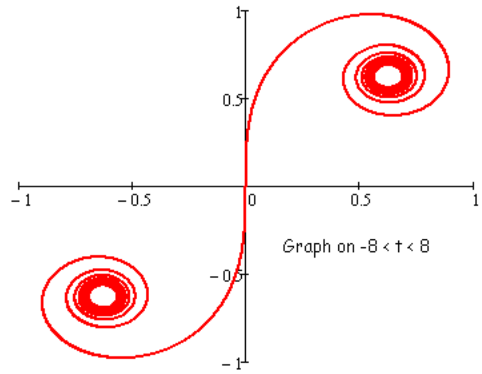
NCAAPMT Calculus Challenge Problem #12 SOLUTION

Consider the relation defined by the parametric equations

$$x(t)=\int_0^t \sin\left(v^2\right)dv$$
$$y(t)=\int_0^t \cos\left(v^2\right)dv$$
.



Use your calculator to sketch the graph of this relation.

Everyone recognized this as a combination of the 2nd Fundamental Theorem of Calculus and parametric form. The curve is known as the Cornu Spiral.

At times the defining equations can be found as
$$x(t)=\int_0^t \sin\left(\frac{v^2}{2}\right)dv \qquad y(t)=\int_0^t \cos\left(\frac{v^2}{2}\right)dv$$
or
$$x(t)=\int_0^t \sin\left(\frac{\pi v^2}{2}\right)dv \qquad y(t)=\int_0^t \cos\left(\frac{\pi v^2}{2}\right)dv$$
.

Other variations are possible.

a) Write the equation of the line tangent to the curve at $t=\sqrt{\pi}$. (1 pt)

We know from the 2nd FTC that $\frac{dx}{dt}=\sin\left(t^2\right)$ and $\frac{dy}{dt}=\cos\left(t^2\right)$. So, at $t=\sqrt{\pi}$, we have

$\frac{dx}{dt}\left(\sqrt{\pi}\right)=\sin\left(\sqrt{\pi}^2\right)=0$ and $\frac{dy}{dt}\left(\sqrt{\pi}\right)=\cos\left(\sqrt{\pi}^2\right)=-1$. Since $\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$, we see that this slope is undefined. There must be a vertical tangent at $t=\sqrt{\pi}$. The location of this tangent is $x\left(\sqrt{\pi}\right)=\int_0^{\sqrt{\pi}}\sin\left(v^2\right)dv$ and $y\left(\sqrt{\pi}\right)=\int_0^{\sqrt{\pi}}\cos\left(v^2\right)dv$. These integrals must be done numerically, since there is no simple antiderivative for either $\sin\left(v^2\right)$ or $\cos\left(v^2\right)$. We approximate this position to be (0.895, 0.663). So the equation of the tangent line is $x=0.895$.

b) Compute the curvature (NCAAPMT Calculus Challenge Problem #5) at $t=\sqrt{\pi}$. (1 pt)

The curvature in parametric form is $k=\frac{|x'y''-y'x''|}{\left((x')^2+(y')^2\right)^{3/2}}$ (all functions of *t*). So, using the 2nd FTC and the chain rule, we have, $k=\frac{\left|\sin(t^2)\left(-2t\cdot\sin(t^2)\right)-\cos(t^2)\left(2t\cdot\cos(t^2)\right)\right|}{\left(\left(\sin(t^2)\right)^2+\left(\cos(t^2)\right)^2\right)^{3/2}}$. This simplifies to $k=\frac{|2t|\cdot\left|\sin^2(t^2)+\cos^2(t^2)\right|}{\left(\left(\sin(t^2)\right)^2+\left(\cos(t^2)\right)^2\right)^{3/2}}=|2t|$, since $\sin^2(t^2)+\cos^2(t^2)=1$. So, at $t=\sqrt{\pi}$ we have $2\sqrt{\pi}$.

c) Find the length of the curve from $t=0$ to $t=\sqrt{\pi}$. (1 pt)

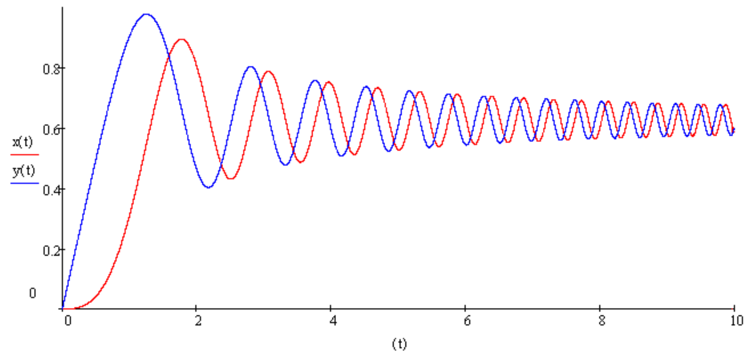
The length of the curve is $L=\int_0^t\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$, so we have $L=\int_0^t\sqrt{\sin^2(t^2)+\cos^2(t^2)}\,dt$. Again we recognize the Pythagorean identity, so $L=\int_0^t\sqrt{1}\,dt=t$. The length of the curve from $t=0$ to $t=\sqrt{\pi}$ is $L=\sqrt{\pi}$

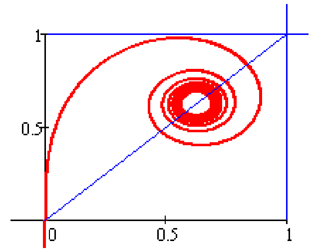
d) Find the relationship between the length of the curve from $t=0$ to $t=a$ and the curvature at $t=a$ for this curve. (1 pt)

Using the work from b) and c), we see that at any position $t=a$, the arc length is $L=a$ and the curvature is $k=|2a|$.

e) As $t\to\infty$, the graph converges to a point. What are the coordinates of this point? (this one is really hard) (1 pt)

We are interested in the long term behavior of this relation. Since we can't find antiderivatives of the integrands, this was intended to be a numerical investigation to approximate the location.

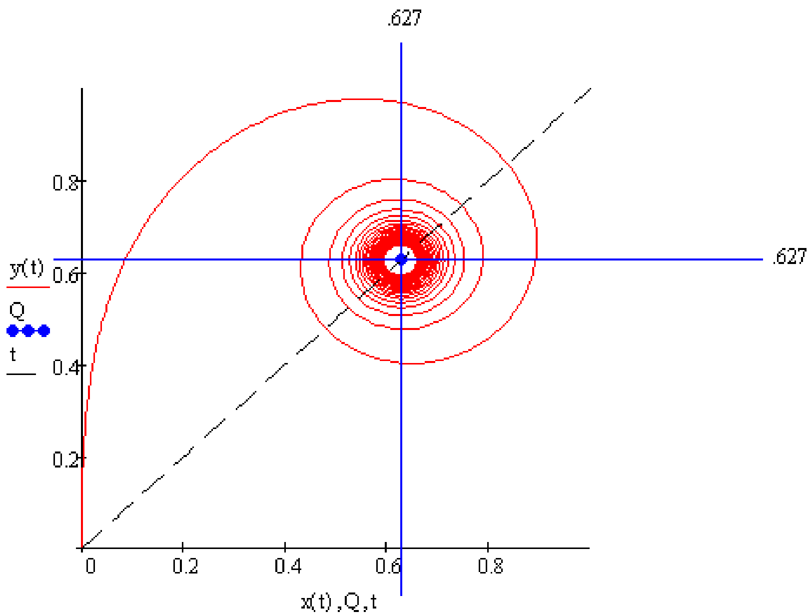




The graphs of the two separate functions indicates that the two functions approach a common value, so we should be looking at a point on $y = x$.

This means that $\int_0^\infty \sin\left(v^2\right) dv = \int_0^\infty \cos\left(v^2\right) dv$.

Playing around with the graph, I estimated the attracting point as (0.627, 0.627)



I was pleased that several of you found the Cornu Spiral and the associated Fresnel Integral on Wikipedia and/or Mathworld: http://en.wikipedia.org/wiki/Fresnel_integral or http://mathworld.wolfram.com/CornuSpiral.html.

These references will tell you (using mathematics that is way beyond introductory calculus) that $\int_0^\infty \sin\left(v^2\right) dv = \int_0^\infty \cos\left(v^2\right) dv = \sqrt{\frac{\pi}{8}}$, and $\sqrt{\frac{\pi}{8}} \approx 0.626657$ so my estimate was actually pretty good.