Early in the season, Pat was hitting fewer than 80% of her free throws. At the end of the season, she was hitting more than 80% of her free throws.

a) Must there have been a time during the season at which she was hitting exactly 80% of her free throws? If so, explain why. If not, give a counter-example. (3 pts)

b) If the answer to a) was yes, find all other values *p* which has this intermediate value property (you can't go from below *p* to above *p* without going through exactly *p*). If the answer to a) was no, are there any values of *p* for which this intermediate value property does hold? If not, explain why there can be no such values. (2 pts)

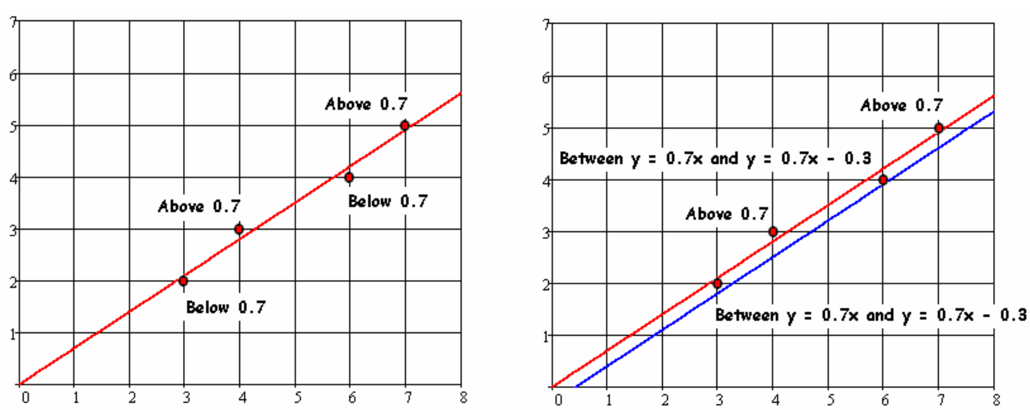
a) Most student groups realized that since the number of attempts is discrete, the Intermediate Value Theorem of Calculus does not apply. However, if the conditions of a theorem are not met (in this case, continuity) that doesn't mean the conclusion will not be satisfied. One cannot go from below 80% to above 80% without passing through exactly 80% after some shot.

To see this you should first try lots of different examples. You shouldn't find one that jumps over 0.8. This suggests that this number has the intermediate value property we are looking for. Clearly every number doesn't. Consider 0.7. We have 2 out of 3 then 3 out of 4 and we jump over 0.7. What is different about 0.7 and 0.8?

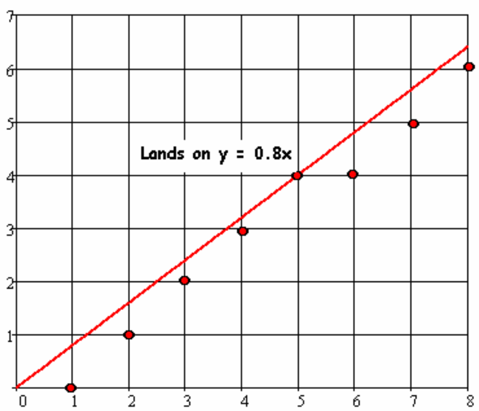
Let *y* be the number of shots made and *x* be the number attempted (here we use x and y to help facilitate graphing. Why did I pick *y* for the number of shots made?). Look at the first quadrant and the line $y = 0.7x$. We see that it is possible to move from a lattice point (point with both coordinates integers) below the line $y = 0.7x$ to a lattice point above the line $y = 0.7x$ without landing on the line. These ratios of integers allow us to jump over 0.7.

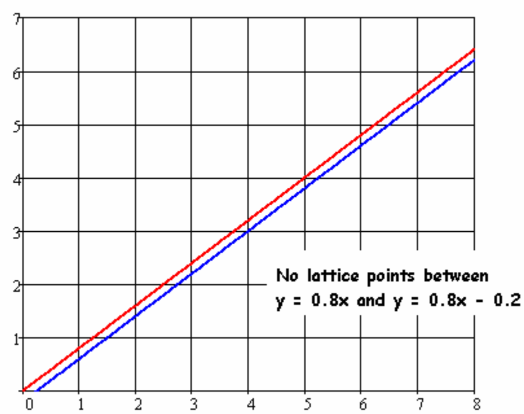
Another way to see what happens is to consider the region between that $\frac{y}{x} < 0.7$ and $\frac{y+1}{x+1} > 0.7$.

Since we require that $\frac{y}{x} < 0.7$ and $\frac{y+1}{x+1} > 0.7$, we need to find integer values of *x* and *y* so that $y < 0.7x$ and $y > 0.7x - 0.3$. If such an ordered pair of integers exists, then we can jump over 0.7. Again, we can see there are numerous such lattice points lying between the lines $y = 0.7x$ and $y = 0.7x - 0.3$ and these are the places where we jump over 0.7.



What is different about $y = 0.8x$? The pattern seen in the graphs below are repeated for all positive integers.





Is there a lattice point between $y = 0.8x$ and $y = 0.8x - 0.2$? The answer is no. Either the interval contains no lattice point or a lattice point lies on one of the two lines (this means that either $\frac{y}{x} = 0.8$ or $\frac{y+1}{x+1} = 0.8$).

Now, lets try to prove this result algebraically. Let a represent the number of shots attempted and m the number of shots made. We know that $m < a$, so $\frac{m}{a} < \frac{m+k}{a+k}$ for all positive integers k. Our questions is, for every choice of m and a ($m < a$ and $\frac{m}{a} < \frac{4}{5}$), is there an integer k that makes $\frac{m+k}{a+k} = \frac{4}{5}$? We will solve this equation for k.

So, $5(m+k) = 4(a+k)$ and $k = 4a - 5m$. If $\frac{m}{a} < \frac{4}{5}$, then k is a positive integer and no matter what the choices for m and a, there is a positive number of additional shots (k) that will produce a success rate of exactly 80%.

If we try the same argument with $\frac{m+k}{a+k} = \frac{7}{10}$, we find that $k = \frac{7a - 10m}{3}$. The value of k is an integer only if 3 divides $7a - 10m$. There are many values of m and a for which 3 does not divide $7a - 10m$. Each of these represents a starting place that will jump over 0.7.

b) To determine the other values that share 0.8's intermediate value property, we need to find integer values of p and q so that there is always an integer k that makes $\frac{m+k}{a+k} = \frac{p}{q}$. Solving for k, we find that $k = \frac{ap - mq}{p - q}$. The value k is an integer only if $p - q$ divides $ap - mq$. Clearly this happens when $p - q = 1$. This means that the ratio of shots made to shots attempted is $\frac{n}{n+1}$. In our example, this is 4 out of 5. Any other ratio of this form will also have this intermediate value property. So, 50%, 80%, 90% do have this property along with 75% and many others.

We haven’t shown that these are the only ratios with this property, and we won’t. That carries us a bit too far from introductory calculus.