NCAAPMT Calculus Challenge Problem #3

In class, you have learned the definition of derivative. We know that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

whenever the limit exists.

The derivative of a function is another function, so we can find its derivative as well. The derivative of the derivative is known as the second derivative and is denoted $f''(x)$.

a) Find a limit definition for $f''(x)$ similar to $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and using only values of *f*. (do not use f' in your limit statement) (3 pts)

SOLUTION

We know that $f'(x)=\frac{d}{dx}f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and that $f''(x)=\frac{d}{dx}f'(x)$.

To make it a little easier to read, define function *g* to be the derivative of *f*, so $g(x)=f'(x)$.

We want $g'(x)=\frac{d}{dx}g(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$.

So $g(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and $g(x+h)=\lim_{h\to 0}\frac{f((x+h)+h)-f(x+h)}{h}$.

So, $f''(x)=\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)=\lim_{h\to 0}\left(\frac{\lim_{h\to 0}\frac{f((x+h)+h)-f(x+h)}{h}-\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}{h}\right)$

We know that the limit of a sum is the sum of the limits, so we have

$$f''(x)=\lim_{h\to 0}\left(\lim_{h\to 0}\frac{\left(f((x+h)+h)-f(x+h)-f(x+h)+f(x)\right)}{h^2}\right)$$

and

$$f''(x)=\lim_{h\to 0}\left(\lim_{h\to 0}\frac{\left(f(x+2h)-2f(x+h)+f(x)\right)}{h^2}\right).$$

Also, $\lim\limits_{h\to 0}\left(\lim\limits_{h\to 0} Q(x)\right)$ is just $\lim\limits_{h\to 0}\left(Q(x)\right)$. The composed limit statements are redundant.

So, we have $f''(x) = \lim\limits_{h\to 0}\left(\dfrac{f(x+2h)-2f(x+h)+f(x)}{h^2}\right)$.

Before we try our definition on a messy cubic, let's try it on something simple, like $f(x)=x^2$ before we go too far down the wrong track. If $f(x)=x^2$, then $\left(\dfrac{f(x+2h)-2f(x+h)+f(x)}{h^2}\right) = \dfrac{(x+2h)^2-2(x+h)^2+x^2}{h^2}$. Simplifying, we have

$$\frac{(x+2h)^2-2(x+h)^2+x^2}{h^2} = \frac{x^2+4hx+4h^2-2x^2-4xh-2h^2+x^2}{h^2} = 2$$

So, $f''(x) = \lim\limits_{h\to 0}\left(\dfrac{f(x+2h)-2f(x+h)+f(x)}{h^2}\right) = \lim\limits_{h\to 0} 2 = 2$ and the second derivative of x^2 is 2. That looks good.

b) Use this limit definition to show that if $f(x)=a_0+a_1x+a_2x^2+a_3x^3$, then $f''(x)=2a_2+6a_3x$. (1 pt)

OK. Now, let's try $f(x)=a_0+a_1x+a_2x^2+a_3x^3$. We need expressions for $f(x+2h)$, $f(x+h)$, and $f(x)$.

$$f(x+2h) = a_0+a_1(x+2h)+a_2(x+2h)^2+a_3(x+2h)^3$$

$$f(x+h) = a_0+a_1(x+h)+a_2(x+h)^2+a_3(x+h)^3$$

$$f(x) = a_0+a_1x+a_2x^2+a_3x^3$$

So, substituting the above expressions into our difference quotient $\dfrac{f(x+2h)-2f(x+h)+f(x)}{h}$ gives the expanded expression

$$\frac{a_0+a_1\left(x+2h\right)+a_2\left(x+2h\right)^2+a_3\left(x+2h\right)^3-2\left(a_0+a_1\left(x+h\right)+a_2\left(x+h\right)^2+a_3\left(x+h\right)^3\right)+a_0+a_1x+a_2x^2+a_3x^3}{h^2}$$

Now, multiply everything out and we have

$$\begin{aligned}\left(\frac{f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)}{h^2}\right) &= \frac{a_0+a_1x+2a_1h+a_2x^2+4a_2h+4a_2h^2+a_3x^3+6a_3hx^2+12a_3h^2x+8a_3h^3}{h^2}\\ &\quad+\frac{-2a_0-2a_1x-2a_1h-2a_2x^2-4a_2h-2a_2h^2-2a_3x^3-6a_3hx^2-6a_3h^2x-2a_3h^3}{h^2}\\ &\quad+\frac{a_0+a_1x+a_2x^2+a_3x^3}{h^2}\end{aligned}$$

Combining like terms, this simplifies to

$$\left(\frac{f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)}{h^2}\right)=\frac{2a_2h^2+6a_3h^2x+6a_3h^3}{h^2}=2a_2+6a_3x+6a_3h\,.$$

So, $f''\left(x\right)=\lim\limits_{h\to 0}\left(\frac{f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)}{h^2}\right)=\lim\limits_{h\to 0}\left(2a_2+6a_3x+6a_3h\right)=2a_2+6a_3x$.

Our limit definition seems to be working.

c) Generalize your limit definition to find the n^{th} derivative, $\frac{d^nf}{dx^n}\left(x\right)$, in terms of f only. (1 pt)

To generalize this result, let's consider the third derivative. We need to replace function g in $\lim\limits_{h\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h}$ with $\left(\frac{f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)}{h^2}\right)$ and see if we can see a pattern.

If you see a pattern, try to explain why the pattern exists or how it is created.

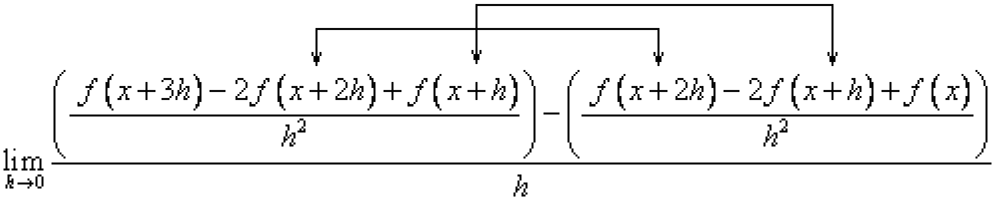
Replacing function g in $\lim\limits_{h\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h}$ with $\left(\frac{f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)}{h^2}\right)$ gives us

$$\lim_{h\to 0}\frac{\left(\dfrac{f(x+3h)-2f(x+2h)+f(x+h)}{h^2}\right)-\left(\dfrac{f(x+2h)-2f(x+h)+f(x)}{h^2}\right)}{h}$$

which simplifies to

$$\lim_{h\to 0}\frac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{h^3}.$$

Notice how the terms in the numerator combined.   The “like terms” in each expression have the same sign and add together.



$$\lim_{h\to 0}\frac{\left(\dfrac{f(x+3h)-2f(x+2h)+f(x+h)}{h^2}\right)-\left(\dfrac{f(x+2h)-2f(x+h)+f(x)}{h^2}\right)}{h}$$

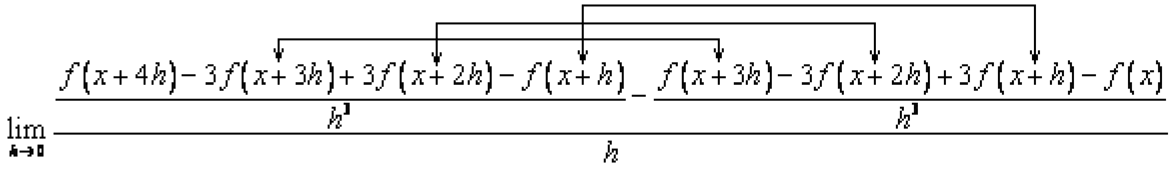
For the 4th derivative, we replace function *g* in $\lim_{h\to 0}\dfrac{g(x+h)-g(x)}{h}$ with $\dfrac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{h^3}$.   This gives

$$\lim_{h\to 0}\frac{\dfrac{f(x+4h)-3f(x+3h)+3f(x+2h)-f(x+h)}{h^3}-\dfrac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{h^3}}{h}$$

which simplifies to

$$\lim_{h\to 0}\frac{f(x+4h)-4f(x+3h)+6f(x+2h)-4f(x+h)+f(x)}{h^4}.$$

Again, look how the terms in the numerator combine.



$$\lim_{h\to 0}\frac{\dfrac{f(x+4h)-3f(x+3h)+3f(x+2h)-f(x+h)}{h^3}-\dfrac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{h^3}}{h}$$

If this looks familiar, you are right. The coefficients combine in the same added way as the elements in Pascal’s triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

These coefficients are known as binomial coefficients, denoted $\binom{n}{k}$. In general, we see that

$$\frac{d^n}{dx^n} f\left(x\right) = \lim_{h \to 0} \frac{\binom{n}{0} f\left(x+nh\right) - \binom{n}{1} f\left(x+\left(n-1\right)h\right) + \binom{n}{2} f\left(x+\left(n-2\right)h\right) - \cdots + \left(-1\right)^n \binom{n}{n} f\left(x\right)}{h^n}.$$