Problem #4 Adding with Derivatives **Due October 29**

We know that the infinite geometric series $1+x+x^2+x^3+x^4+\cdots=\dfrac{1}{1-x}$ whenever $|x|<1$.

The expression $\dfrac{1}{1-x}$ is known as the *closed form* of the series. Using the closed form, we know that $1+\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^4+\cdots=\dfrac{1}{1-\left(\frac{1}{3}\right)}=\dfrac{3}{2}$.

a) First show that if $1+x+x^2+x^3+\cdots=\dfrac{1}{1-x}$ whenever $|x|<1$, then $1+2x+3x^2+4x^3+\cdots=\dfrac{1}{(1-x)^2}$ whenever $|x|<1$. Find $1+2\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+4\left(\frac{1}{3}\right)^3+\cdots$ (1 pt)

We know that $1+x+x^2+x^3+x^4+\cdots=\dfrac{1}{1-x}$ whenever $|x|<1$, so by differentiating both sides of this equation, we have $1+2x+3x^2+4x^3+\cdots=\dfrac{1}{(1-x)^2}$ whenever $|x|<1$. To determine the sum of the infinite series $1+2\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+4\left(\frac{1}{3}\right)^3+5\left(\frac{1}{3}\right)^4+\cdots$ we just evaluate the closed form at $x=\frac{1}{3}$. The closed form for a series is known as the *generating function* for that series, so we typically denote it as $G(x)$. If $G(x)=\dfrac{1}{(1-x)^2}$, then $G\left(\frac{1}{3}\right)=\dfrac{1}{\left(1-\frac{1}{3}\right)^2}=\dfrac{9}{4}$.

So, $1+2\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+4\left(\frac{1}{3}\right)^3+5\left(\frac{1}{3}\right)^4+\cdots=\frac{9}{4}$.

b) Find a closed form for $x+2x^2+3x^3+4x^4+5x^5+\cdots$. Modify your expression to find the sum of $1+4\left(\frac{1}{3}\right)+9\left(\frac{1}{3}\right)^2+16\left(\frac{1}{3}\right)^3+25\left(\frac{1}{3}\right)^4+\cdots$ (1 pt)

We know that $1+x+x^2+x^3+x^4+\cdots=\dfrac{1}{1-x}$ and $1+2x+3x^2+4x^3+\cdots=\dfrac{1}{(1-x)^2}$. How do we find a closed form for $x+2x^2+3x^3+4x^4+5x^5+\cdots$? If we multiply our second series by x $(x\neq 0)$, we have $x+2x^2+3x^3+4x^4+\cdots=\dfrac{x}{(1-x)^2}$ whenever $|x|<1$.

How do we use this to sum $1+4\left(\frac{1}{3}\right)+9\left(\frac{1}{3}\right)^2+16\left(\frac{1}{3}\right)^3+25\left(\frac{1}{3}\right)^4+\cdots$? We need to find the closed form for $1+4x+9x^2+16x^3+25x^4+\cdots$ and evaluate at $x=\frac{1}{3}$.

Notice that $1+4x+9x^2+16x^3+25x^4+\cdots$ is the derivative of $x+2x^2+3x^3+4x^4+5x^5\cdots$, so its closed form should be the derivative of $\dfrac{x}{(1-x)^2}$.

If $x+2x^2+3x^3+4x^4+\cdots=\dfrac{x}{(1-x)^2}$ whenever $|x|<1$, then

$$1+4x+9x^2+16x^3+25x^4+\cdots=\frac{(1+x)^2(1)-x(2)(x-1)^1(-1)}{(1-x)^4}=\frac{1+x}{(1-x)^3}\text{ whenever }|x|<1.$$

So, now $G(x)=\dfrac{1+x}{(1-x)^3}$. Evaluating at $x=\frac{1}{3}$ give us $G\left(\frac{1}{3}\right)=\dfrac{1+\frac{1}{3}}{\left(1-\frac{1}{3}\right)^3}=\dfrac{9}{2}$.

$$1+4\left(\tfrac{1}{3}\right)+9\left(\tfrac{1}{3}\right)^2+16\left(\tfrac{1}{3}\right)^3+25\left(\tfrac{1}{3}\right)^4+\cdots=\frac{9}{2}.$$

c) $1-4\left(\frac{1}{3}\right)+9\left(\frac{1}{3}\right)^2-16\left(\frac{1}{3}\right)^3+25\left(\frac{1}{3}\right)^4+\cdots$ (1 pt)

The coefficients are squares as before, but they alternate in sign. We know that
$1+4x+9x^2+16x^3+25x^4+\cdots=\dfrac{1+x}{(1-x)^3}$ whenever $|x|<1$.

If we replace x with $-x$, we have $1+4(-x)+9(-x)^2+16(-x)^3+25(-x)^4+\cdots=\dfrac{1+(-x)}{\left(1-(-x)\right)^3}$ whenever $|(-x)|<1$. This is just $1-4x+9x^2-16x^3+25x^4+\cdots=\dfrac{1-x}{(1+x)^3}$ whenever $|x|<1$.

So, $G(x)=\dfrac{1-x}{(1+x)^3}$. Evaluating at $x=\frac{1}{3}$ give us $G\left(\frac{1}{3}\right)=\dfrac{1-\frac{1}{3}}{\left(1+\frac{1}{3}\right)^3}=\dfrac{9}{32}$.

$$1+4\left(\tfrac{1}{3}\right)+9\left(\tfrac{1}{3}\right)^2+16\left(\tfrac{1}{3}\right)^3+25\left(\tfrac{1}{3}\right)^4+\cdots=\frac{9}{32}.$$

d) $1\left(\frac{1}{3}\right)^2+8\left(\frac{1}{3}\right)^4+27\left(\frac{1}{3}\right)^6+64\left(\frac{1}{3}\right)^8+125\left(\frac{1}{3}\right)^{10}+\cdots$ (1 pt)

OK. Now they are starting to get tricky. We need to find the closed form for the series
$1x^2+8x^4+27x^6+64x^8+125x^{10}+\cdots$. The coefficients are cubes and the powers jump by two and they don't start with a constant.

First, let's deal with the powers. We know that $1+4x+9x^2+16x^3+25x^4+\cdots=\dfrac{1+x}{(1-x)^3}$ whenever $|x|<1$. If we differentiate now, we get $4+18x+48x^2+\cdots$. That's not what we want.

If we multiply both sides by *x*, we have $x+4x^2+9x^3+16x^4+25x^5+\cdots=\dfrac{x(1+x)}{(1-x)^3}$.

Now differentiate both sides of the equation (and be careful with the quotient rule).

So, $1+8x+27x^2+64x^3+125x^4+\cdots=\dfrac{x^2+4x+1}{(1-x)^4}$. That gets the coefficients right. Now for the powers.

If $1+8x+27x^2+64x^3+125x^4+\cdots=\dfrac{x^2+4x+1}{(1-x)^4}$, whenever $|x|<1$, then by replacing *x* with x^2 we have
$1+8x^2+27x^4+64x^6+125x^8+\cdots=\dfrac{x^4+4x^2+1}{\left(1-x^2\right)^4}$, whenever $\left|x^2\right|<1$ We are close to what we want. Now multiply both sides of the equation by x^2. So,

$1x^2+8x^4+27x^6+64x^8+125x^{10}+\cdots=\dfrac{x^6+4x^4+x^2}{\left(1-x^2\right)^4}$, whenever $\left|x^2\right|<1$. Since $\left(\frac{1}{3}\right)^2<1$ we are OK. So, $G(x)=\dfrac{x^6+4x^4+x^2}{\left(1-x^2\right)^4}$ and $G\left(\frac{1}{3}\right)=\dfrac{\left(\frac{1}{3}\right)^6+4\left(\frac{1}{3}\right)^4+\left(\frac{1}{3}\right)^2}{\left(1-\left(\frac{1}{3}\right)^2\right)^4}=\dfrac{531}{2048}$.

$$1\left(\tfrac{1}{3}\right)^2+8\left(\tfrac{1}{3}\right)^4+27\left(\tfrac{1}{3}\right)^6+64\left(\tfrac{1}{3}\right)^8+125\left(\tfrac{1}{3}\right)^{10}+\cdots=\frac{531}{2048}.$$

e) $1\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^3+\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)^4+\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^5+\cdots$ (1 pt)

Now we are just being mean!
What is the generating function for $1x+\left(\frac{1}{2}\right)x^2+\left(\frac{1}{3}\right)x^3+\left(\frac{1}{4}\right)x^4+\left(\frac{1}{5}\right)x^5+\cdots$? We need to think backwards on this one. We must notice that $1+x+x^2+x^3+x^4+x^5+\cdots$ is the derivative of

$1x+\left(\tfrac{1}{2}\right)x^2+\left(\tfrac{1}{3}\right)x^3+\left(\tfrac{1}{4}\right)x^4+\left(\tfrac{1}{5}\right)x^5+\cdots$. So, $\frac{1}{1-x}$ must be the derivative of the generating function we are after. What function has $\frac{1}{1-x}$ as its derivative? Well, $\frac{d}{dx}\left(-\ln\left(1-x\right)\right)=\frac{-1}{1-x}\left(-1\right)=\frac{1}{1-x}$ (don't forget the chain rule).

So, does $1x+\left(\tfrac{1}{2}\right)x^2+\left(\tfrac{1}{3}\right)x^3+\left(\tfrac{1}{4}\right)x^4+\left(\tfrac{1}{5}\right)x^5+\cdots=-\ln\left(1-x\right)$, whenever $|x|<1$. Notice that the values match at $x=0$. So, $G\left(x\right)=-\ln\left(1-x\right)$ and $G\left(\tfrac{1}{3}\right)=-\ln\left(\tfrac{2}{3}\right)$. The sum is approximated by 0.405465.

Super Challenge: $0+1\left(\tfrac{1}{3}\right)+1\left(\tfrac{1}{3}\right)^2+2\left(\tfrac{1}{3}\right)^3+3\left(\tfrac{1}{3}\right)^4+5\left(\tfrac{1}{3}\right)^5+8\left(\tfrac{1}{3}\right)^6+13\left(\tfrac{1}{3}\right)^7\cdots$ (the coefficients are the Fibonacci numbers) (0 pts, this one is just for fun and is much, much harder)

No one is supposed to have gotten this one. I just wanted you to see how these kinds of things are done.

To find a generating function like this one, we assume one exists and try to find its structure. In this case we say that $G\left(x\right)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$ and we know that the coefficients are defined by the Fibonacci recursion $a_0=0,\ a_1=1,$ and $a_{n+2}=a_{n+1}+a_n$ (sometimes it is convenient to start with 1, 1, 2, 3, 5, 8, … and other times 0, 1, ,1, 2, 3, 5, 8, …). In this case we want the first term to be 0.

So, if $a_{n+2}=a_{n+1}+a_n$, we multiply both sides of this equation by x^n and evaluate for $n=0,1,2,3\ldots$.

$n=0,1,2,3\ldots$	$a_{n+2}\left(x^n\right)=a_{n+1}\left(x^n\right)+a_n\left(x^n\right)$
$n=0$	$a_2=a_1+a_0$
$n=1$	$a_3\left(x\right)=a_2\left(x\right)+a_1\left(x\right)$
$n=2$	$a_4\left(x^2\right)=a_3\left(x^2\right)+a_2\left(x^2\right)$
$n=3$	$a_5\left(x^3\right)=a_4\left(x^3\right)+a_3\left(x^3\right)$
$n=4$	$a_6\left(x^4\right)=a_5\left(x^3\right)+a_4\left(x^4\right)$
$n=5$	$a_7\left(x^5\right)=a_6\left(x^5\right)+a_5\left(x^5\right)$

Now, add up the equations in the second column. We have

$$a_2 + a_3x + a_4x^2 + a_5x^3 + a_6x^4 + \cdots = \left(a_1 + a_2x + a_3x^2 + a_4x^3 \cdots\right) + \left(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\right)$$

We have three series that are each a modification of our proposed generating function $G(x)$.

The left side of the equation is $G(x)$ with the first two terms gone and some *x*'s divided out. We can write

$$a_2 + a_3x + a_4x^2 + a_5x^3 + a_6x^4 + a_7x^5 + \cdots = \frac{G(x) - a_0 - a_1x}{x^2}.$$

Make sure you believe that statement before proceeding.

The right hand side has two variations of $G(x)$.
The first is

$$a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 \cdots = \frac{G(x) - a_0}{x}$$

while the second is just

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = G(x).$$

So, $a_2 + a_3x + a_4x^2 + a_5x^3 + a_6x^4 + \cdots = \left(a_1 + a_2x + a_3x^2 + a_4x^3 \cdots\right) + \left(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\right)$ can be rewritten as

$$\frac{G(x) - a_0 - a_1x}{x^2} = \frac{G(x) - a_0}{x} + G(x).$$

Solve for $G(x)$.

So, $G(x) - a_0 - a_1x = xG(x) - a_0x + x^2G(x)$ and $G(x) = \dfrac{a_0x - a_0 - a_1x}{x^2 + x - 1}$. We also know that $a_0 = 0, a_1 = 1,$ so

$$G(x) = \frac{-x}{x^2 + x - 1}.$$

Finally, we have $G\left(\frac{1}{3}\right) = \dfrac{-\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right) - 1} = \dfrac{3}{5}$.

The question of for what values $G(x) = \dfrac{-x}{x^2 + x - 1}$ actually represents the infinite series $0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 \cdots$ is an important one that we cannot answer with the mathematics that you presently know. It turns out that

$$0+1x+1x^2+2x^3+3x^4+5x^5+8x^6+13x^7\cdots=\frac{-x}{x^2+x-1}, \text{ for } |x|<\frac{\sqrt{5}-1}{2}.$$

To convince yourself that $G(x)=\dfrac{-x}{x^2+x-1}$ actually generates the Fibonacci series, just do the long division and see how the terms are created.