

Problem #4 Adding with Derivatives**Due October 29**

In Algebra II, you learned to sum an infinite geometric series.

$$1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x} \text{ whenever } |x| < 1.$$

The expression $\frac{1}{1-x}$ is known as the *closed form* of the infinite geometric series

$$1 + x + x^2 + x^3 + x^4 + \dots.$$

By using the closed form, we know that $1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \dots = \frac{1}{1 - \left(\frac{1}{3}\right)} = \frac{3}{2}$.

In this challenge problem you will use the geometric series and its closed form and the rules for differentiation combined with algebraic manipulations to find the sum of several difficult infinite series.

a) First show that if $1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x}$ whenever $|x| < 1$, then

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \frac{1}{(1-x)^2} \text{ whenever } |x| < 1.$$

Now, find the sum $1 + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^4 + \dots$ (1 pt)

Use a similar method to find the following sums (for full credit, *be sure to show how you modified the geometric series to find the appropriate sum*).

b) Find a closed form for $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots$.

Modify your expression to find the sum of $1 + 4\left(\frac{1}{3}\right) + 9\left(\frac{1}{3}\right)^2 + 16\left(\frac{1}{3}\right)^3 + 25\left(\frac{1}{3}\right)^4 + \dots$ (1 pt)

Got the idea? Now use derivatives to find the sums of the following infinite series. Show your work.

c) $1 - 4\left(\frac{1}{3}\right) + 9\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right)^3 + 25\left(\frac{1}{3}\right)^4 + \dots$ (1 pt)

d) $1\left(\frac{1}{3}\right)^2 + 8\left(\frac{1}{3}\right)^4 + 27\left(\frac{1}{3}\right)^6 + 64\left(\frac{1}{3}\right)^8 + 125\left(\frac{1}{3}\right)^{10} + \dots$ (1 pt)

e) $1\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)^4 + \left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^5 + \dots$ (1 pt)

Super Challenge: $0 + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right)^4 + 5\left(\frac{1}{3}\right)^5 + 8\left(\frac{1}{3}\right)^6 + 13\left(\frac{1}{3}\right)^7 \dots$ (the coefficients are the Fibonacci numbers) (0 pts, this one is just for fun and is much, much harder)