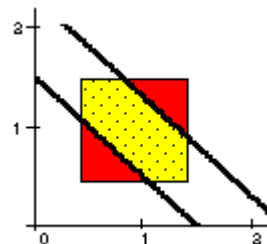


Areas as Probabilities

Each problem below uses areas to determine probabilities of various occurrences. For example:

Given a pair of numbers that both round to 1, and assuming all pairs are equally likely, find the probability that their sum rounds to 2. For example, both 1.4 and 1.3 round to 1, but their sum 2.7 does not.

Sample Solution: Consider the two numbers x and y , with $0.5 \leq x < 1.5$ and $0.5 \leq y < 1.5$. These are the numbers that round to 1. This gives us a sample space of a square. All points in the square represent possible values of x and y . If $x + y \geq 1.5$ and $x + y < 2.5$, then the sum would round to 2. So, we want to find what fraction of the square $0.5 \leq x < 1.5$ and $0.5 \leq y < 1.5$ contains those points. We want to know the area inside the square between these two lines.



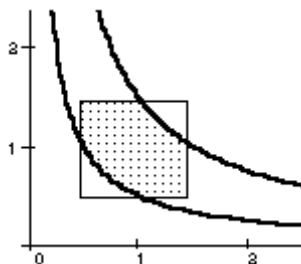
From the figure above, we see that $P(1+1=1) = \frac{1}{8}$, $P(1+1=2) = \frac{3}{4}$, and $P(1+1=3) = \frac{1}{8}$.

The Problems below are similar in nature:

1 Given a pair of numbers that both round to 1, and assuming all pairs are equally likely

a) What is the probability their product rounds to 1? (1 pt)

In this case we require $xy \geq 0.5$ and $xy < 1.5$.



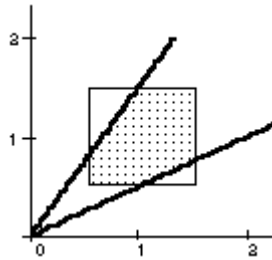
We need to find the area between the curves and inside the box. Probably the easiest way to do this is to find the area of the two unshaded regions. The first is $\int_0^{0.5} \frac{0.5}{x} dx = \frac{\ln 2}{2} - \frac{1}{4} \approx 0.097$. The second is

$\int_1^{1.5} \frac{1.5}{x} dx = \frac{3}{4} - \frac{3}{2} \ln\left(\frac{3}{2}\right) \approx 0.142$. The middle region that we want is (there are lots of equivalent forms for logarithmic expressions, so yours may differ from mine)

$1 - \left(\frac{\ln 2}{2} - \frac{1}{4}\right) - \left(\frac{3}{4} - \frac{3}{2} \ln\left(\frac{3}{2}\right)\right) = \frac{1}{2} + \frac{3}{2} \ln(3) - 2 \ln(2) \approx 0.762$. So, about 76% of the time, the product of two numbers each of which rounds to one will also round to one. So, does this mean that $1 \cdot 1 = 1$ about 76% of the time?

b) What is the probability their quotient rounds to 1? (1 pt)

For the final case we require $\frac{y}{x} \geq 0.5$ and $\frac{y}{x} < 1.5$.

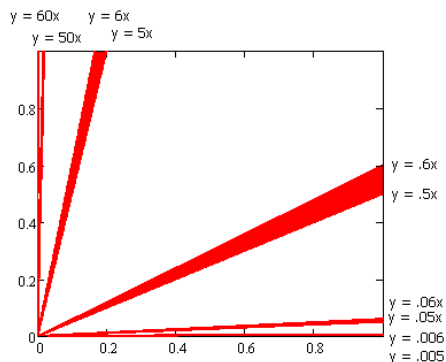
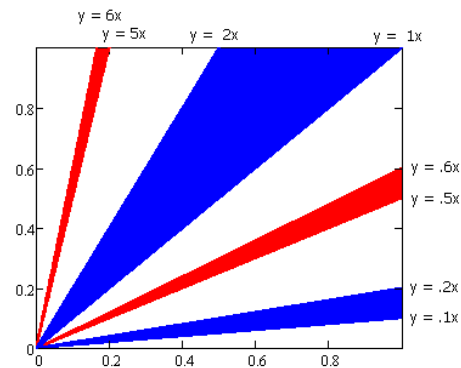


Again, we want the area of the middle region, which is 1 minus the two unshaded triangles. These areas are $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$ for the bottom triangle and $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{3}{16}$ for the top triangle. So, the area we want is $1 - \frac{1}{16} - \frac{3}{16} = \frac{3}{4}$. So, 75% of the time, the quotient of two number each of which rounds to one will also round to one.

2) Suppose x and y are each chosen at random from $(0, 1)$ with all values being equally likely. Compute the ratio $R = \frac{y}{x}$. What is the probability that the first non-zero digit in R (written as a decimal) is a 5? For example, 52.1772 and .053281 both have 5 as the first non-zero digit. (2 pt)

This problem was a bit more subtle than most students realized. If you use your calculator to make a list of 100 ratios of rand/rand, you will see that the vast majority of them have a first non-zero digit of 1 and very few have a first non-zero digit of 9. The distribution of first non-zero digits is very uneven.

The figure at right illustrates the non-uniformity. The blue regions begin with a first non-zero digit of 1 and the red begin with 5. The areas of the two triangles below $y = x$ are the same. If the ratio is less than one, then the first non-zero digits are uniformly distributed. However, if the ratio is larger than one, things get more interesting. Notice how much larger the region between $y = x$ and $y = 2x$ (in blue) is than the region between $y = 5x$ and $y = 6x$ (in red). Of course, there are an infinite number of triangular regions to consider.



We are interested in the first non-zero digit of 5. If $R = \frac{y}{x} < 1$, then the regions between the lines $y = .5x$ and $y = .6x$, $y = .05x$ and $y = .06x$, between $y = .005x$ and $y = .006x$, etc. are all triangles with a height of 1 and a base that is the length of the segment on the boundary of the square, which is 0.1, 0.01, and 0.001 units respectively. So, the total areas $\frac{1}{2}bh$ for this lower portion are

$$A_L = \left(\frac{1}{2}\right)(1)\left(\frac{1}{10}\right) + \left(\frac{1}{2}\right)(1)\left(\frac{1}{100}\right) + \left(\frac{1}{2}\right)(1)\left(\frac{1}{1000}\right) + \left(\frac{1}{2}\right)(1)\left(\frac{1}{10000}\right) + \dots$$

This is an infinite geometric series with a sum of $\frac{\frac{1}{20}}{1 - \frac{1}{10}} = \frac{1}{18}$.

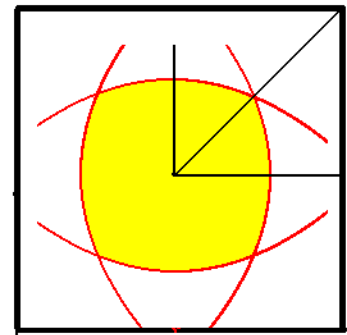
The areas for the upper regions with $R = \frac{y}{x} > 1$ are a bit more challenging. In each case, we have a triangle with an altitude of 1 and a base the length of $\frac{1}{5} - \frac{1}{6}$, $\frac{1}{50} - \frac{1}{60}$, $\frac{1}{500} - \frac{1}{600}$, ... These lengths are $\frac{1}{30}$, $\frac{1}{300}$, $\frac{1}{3000}$, ... These areas also form an infinite geometric series which we can sum.

$$A_u = \left(\frac{1}{2}\right)(1)\left(\frac{1}{30}\right) + \left(\frac{1}{2}\right)(1)\left(\frac{1}{300}\right) + \left(\frac{1}{2}\right)(1)\left(\frac{1}{3000}\right) + \left(\frac{1}{2}\right)(1)\left(\frac{1}{30000}\right) + \dots = \frac{\frac{1}{60}}{1 - \frac{1}{10}} = \frac{1}{54}$$

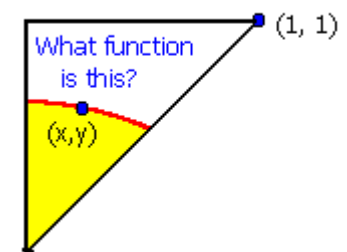
So, the total area occupied by ratios whose first non-zero digit is 5 is $A_L + A_u = \frac{1}{18} + \frac{1}{54} = \frac{2}{27}$.

3) What is the probability that a dart, hitting a square board at random, lands nearer the center than the edge? (1 pts)

The boundary that is equidistant from the center and the edge is not a straight line, but curved as in the figure at right. The yellow region is closer to the center than the edge and the red boundary is the key to answering the question. For simplicity, let the corners of the square be (1, 1), (1, -1), (-1, 1), and (-1, -1) with the center at the origin. The figure is not drawn to scale, but hopefully, you get the idea.



The area we want is really eight copies of the same little piece, shown below at right. We can use integration to find the area if we know the function that acts as the upper boundary (in red).



We want to find the ordered pairs (x, y) that are the same distance from the top of the square ($y = 1$) as the center $(0, 0)$. So, we have the defining equation, $1 - y = \sqrt{x^2 + y^2}$. This can be simplified to $(1 - y)^2 = x^2 + y^2$ and $1 - 2y = x^2$ so $y = \frac{1 - x^2}{2}$. The area between

this curve and $y = x$ between $x = 0$ and $x = \frac{1}{2}$ is $\int_0^{\frac{1}{2}} \frac{1 - x^2}{2} - x \, dx$.

The value of this integral is $\frac{4\sqrt{2} - 5}{6}$. The total area of the triangle is one half, so the probability is

$$\frac{\left(\frac{4\sqrt{2} - 5}{6}\right)}{\left(\frac{1}{2}\right)} = \frac{4\sqrt{2} - 5}{3} \approx 0.219. \text{ About 22\% of the time, a dart, hitting a square board at random,}$$

lands nearer the center than the edge.