Calculus Challenge #10 **Solution**

Fourier Series

A Fourier series is an infinite trigonometric series of the form

$$F(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + a_3\cos(3x) + b_3\sin(3x) + \dots$$

Fourier series can be used to approximate a function over a specific interval. If you have studied Taylor series, you know that they use derivatives to approximate a function at a point. To approximate a function with a Fourier series on an interval, you need an integral.

1. Simplify $\cos(nx+mx)+\cos(nx-mx)$ using the sum and difference identities from trigonometry and use it to evaluate $\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx$ when $m \neq n$ and when $m = n$ (*this problem should have stated that m and n are positive integers*).

We know $\cos(nx+mx) = \cos(nx)\cos(mx) - \sin(nx)\sin(mx)$ and $\cos(nx-mx) = \cos(nx)\cos(mx) + \sin(nx)\sin(mx)$, so $\cos(nx+mx)+\cos(nx-mx) = 2\cos(nx)\cos(mx)$.

$\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x + \cos(n-m)x\,dx$. We know that $\int_{-\pi}^{\pi}\cos(kx)\,dx = 0$ for any integer *k*, so, when *m* and *n* are integers and $m \neq n$, this integral is zero. This is the situation we see repeatedly in the Fourier series analysis.

If $m = n$, then $\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi}\cos^2(nx)\,dx$. We can use half angle formulas or integration by parts to see that $\int_{-\pi}^{\pi}\cos^2(nx)\,dx = \pi$.

2. An even Fourier series has only the cosine terms, and can be used to approximate an even function, so $F_E(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \dots$. In this challenge, we will develop an even Fourier approximation for some general even function f.

a) Use the result from 1) above to find the value of a_0 in terms of $\int_{-\pi}^{\pi} f(x)\,dx$ if

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \ldots\,dx.$$

Since all of the integrals involving the cosine are zero, $\int_{-\pi}^{\pi} \cos(kx)\,dx = 0$, we have only

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx = a_0(2\pi), \text{ so } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

b) Now, find the value of a_1 in terms of $\int_{-\pi}^{\pi} \cos(x) f(x)\,dx$ if

$$\int_{-\pi}^{\pi} \cos(x) f(x)\,dx = \int_{-\pi}^{\pi} a_0\cos(x) + a_1\cos^2(x) + a_2\cos(x)\cos(2x) + a_3\cos(x)\cos(3x) + \ldots\,dx$$

From part 1), we found that all of the terms except the squared term is zero, so we have only

$$\int_{-\pi}^{\pi} \cos(x) f(x)\,dx = \int_{-\pi}^{\pi} a_1\cos^2(x)\,dx = a_1(\pi). \text{ So, } a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x) f(x)\,dx.$$

By multiplying our original function f by various cosines, we can find the other coefficients.

c) Generalize to find the value of a_n in terms of $\int_{-\pi}^{\pi} \cos(nx) f(x)\,dx$ if

$$\int_{-\pi}^{\pi} \cos(nx) f(x)\,dx = \int_{-\pi}^{\pi} a_0\cos(nx) + a_1\cos(nx)\cos(x) + a_2\cos(nx)\cos(2x) + a_3\cos(nx)\cos(3x) + \ldots\,dx$$

It might help to look at $n = 2$ and $n = 3$ first. This result gives us a rule for finding the coefficients to approximate any even function on $[-\pi, \pi]$.

For $n = 2$, we have

$$\int_{-\pi}^{\pi} \cos(2x) f(x)\,dx = \int_{-\pi}^{\pi} a_0\cos(2x) + a_1\cos(2x)\cos(x) + a_2\cos^2(2x) + a_3\cos(2x)\cos(3x) + \ldots\,dx$$ and

again, all but the squared term is zero on the right side of the equation. So,

$$\int_{-\pi}^{\pi} \cos(2x) f(x)\,dx = a_2\int_{-\pi}^{\pi} \cos^2(2x)\,dx = a_2(\pi) \text{ and } a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2x) f(x)\,dx.$$

All other terms work the same way. Only the squared term will be non-zero, and its value will always be π (except for a_0).

So, the general form will be $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx$ for $n = 1, 2, 3, \ldots$.

3. If $f(x) = e^{-x^2}$ on the interval $[-\pi, \pi]$, use an even Fourier series and numerical integration on our calculators to determine the coefficients $a_0, a_1, a_2, a_3, a_4,$ and a_5.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x^2}\,dx \approx 0.28209$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x)e^{-x^2}\,dx \approx 0.43940$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2x)e^{-x^2}\,dx \approx 0.20755$$

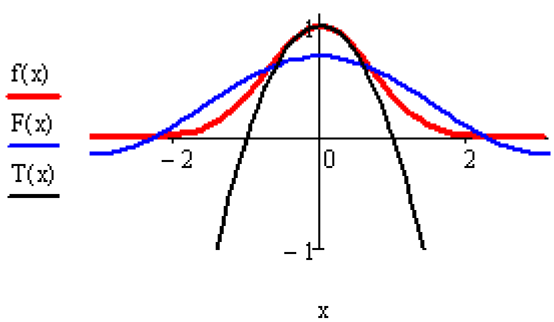
$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(3x)e^{-x^2}\,dx \approx 0.05947$$

$$a_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(4x)e^{-x^2}\,dx \approx 0.01033$$

$$a_5 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(5x)e^{-x^2}\,dx \approx 0.00109$$

In the graphs below, $f(x) = e^{-x^2}$ is shown in **Red**. The Fourier approximation is in **Blue**, and the Taylor approximation is in **Black**. To make a least squares comparison of the Taylor polynomial approximation on this interval to the Fourier approximation, compare the values of $\int_{-\pi}^{\pi}\left(f(x) - F_{E_n}(x)\right)^2 dx$ and $\int_{-\pi}^{\pi}\left(f(x) - T_n(x)\right)^2 dx$ as the number of terms increases.

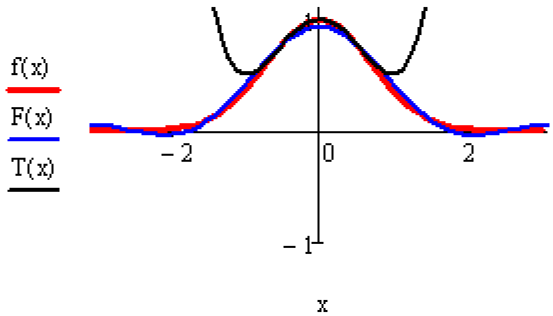
Consider the graphs of $f(x) = e^{-x^2}$ on $[-\pi, \pi]$ with $F_{E_1}(x) = a_0 + a_1\cos(x)$ and $T_1(x) = 1 - x^2$



$$\int_{-\pi}^{\pi} (f(x) - F(x))^2\,dx = 0.14677881$$

$$\int_{-\pi}^{\pi} (f(x) - T(x))^2\,dx = 86.82990801$$

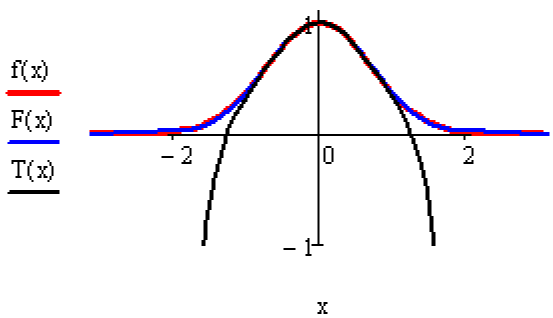
$$F_{E_2}(x) = a_0 + a_1\cos(x) + a_2\cos(2x) \text{ and } T_2(x) = 1 - x^2 + \frac{x^4}{2}$$



$$\int_{-\pi}^{\pi} (f(x) - F(x))^2\, dx = 0.01144955$$

$$\int_{-\pi}^{\pi} (f(x) - T(x))^2\, dx = 1001.03045274$$

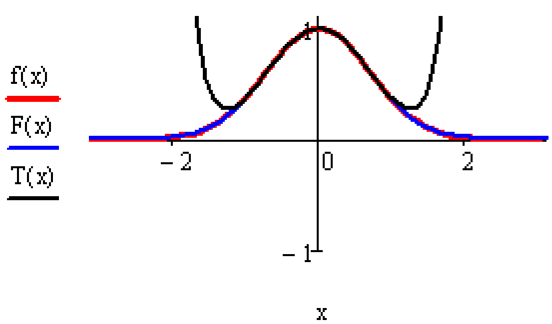
$$F_{E_3}(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) \text{ and } T_3(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$$



$$\int_{-\pi}^{\pi} (f(x) - F(x))^2\, dx = 0.00033898$$

$$\int_{-\pi}^{\pi} (f(x) - T(x))^2\, dx = 6416.16982181$$

$$F_{E_4}(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + a_4\cos(4x) \text{ and } T_4(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}$$



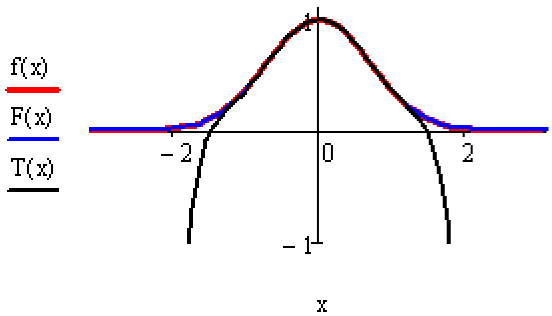
$$\int_{-\pi}^{\pi} (f(x) - F(x))^2\, dx = 0.00000376$$

$$\int_{-\pi}^{\pi} (f(x) - T(x))^2\, dx = 25822.81832517$$

and

$F_{E_4}(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + a_4\cos(4x) + a_5\cos(5x)$ and

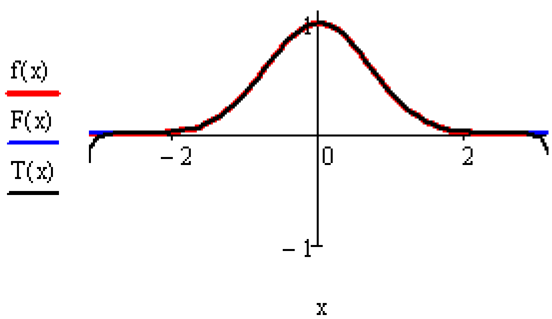
$T_5(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120}$



$$\int_{-\pi}^{\pi} (f(x) - F(x))^2\,dx = 0.00000001$$

$$\int_{-\pi}^{\pi} (f(x) - T(x))^2\,dx = 71126.02152921$$

It is not until n = 25 that the two squared error terms become commensurate.



$$\int_{-\pi}^{\pi} (f(x) - F(x))^2\,dx = 0$$

$$\int_{-\pi}^{\pi} (f(x) - T(x))^2\,dx = 0.00100205$$

Which approximation seems to do a better job on of approximating $f(x) = e^{-x^2}$ on the interval $[-\pi, \pi]$?

The Taylor approximation is always better around x = 0, but the Fourier approximation is clearly better by the least squares criterion (on this interval).