Calculus Challenge #10 **Solutions Due March 11**

Fourier Series

1. Simplify $\cos(nx+mx)+\cos(nx-mx)$ using the sum and difference identities from trigonometry and use it to evaluate $\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx$ when $m\neq n$ and when $m=n$.

A Fourier series is an infinite trigonometric series of the form

$$F(x)=a_0+a_1\cos(x)+b_1\sin(x)+a_2\cos(2x)+b_2\sin(2x)+a_3\cos(3x)+b_3\sin(3x)+\ldots$$

Fourier series can be used to approximate a function over a specific interval. (If you have studied Taylor series, you know that they use derivatives to approximate a function at a point. To approximate a function with a Fourier series on an interval, you need an integral. If you haven't studied Taylor series yet, that's OK.)

2. An even Fourier series has only the cosine terms, and can be used to approximate an even function, so $F_E(x)=a_0+a_1\cos(x)+a_2\cos(2x)+a_3\cos(3x)+\ldots$. In this challenge, we will develop an even Fourier approximation for some general even function f.

a) Use the result from 1) above to find the value of a_0 in terms of $\int_{-\pi}^{\pi} f(x)\,dx$ if

$$\int_{-\pi}^{\pi} f(x)\,dx=\int_{-\pi}^{\pi} a_0+a_1\cos(x)+a_2\cos(2x)+a_3\cos(3x)+\ldots dx$$

b) Now, find the value of a_1 in terms of $\int_{-\pi}^{\pi}\cos(x)f(x)\,dx$ if

$$\int_{-\pi}^{\pi}\cos(x)f(x)\,dx=\int_{-\pi}^{\pi} a_0\cos(x)+a_1\cos^2(x)+a_2\cos(x)\cos(2x)+a_3\cos(x)\cos(3x)+\ldots dx$$

By multiplying our original function f by cosines, we can find the other coefficients.

c) Generalize to find the value of a_n in terms of $\int_{-\pi}^{\pi}\cos(nx)f(x)\,dx$ if

$$\int_{-\pi}^{\pi}\cos(nx)f(x)\,dx=\int_{-\pi}^{\pi} a_0\cos(nx)+a_1\cos(nx)\cos(x)+a_2\cos(nx)\cos(2x)+a_3\cos(nx)\cos(3x)+\ldots dx$$

It might help to look at $n=2$ and $n=3$ first. This result gives us a rule for finding the coefficients to approximate any even function on $[-\pi,\pi]$.

3. If $f(x)=e^{-x^2}$ on the interval $[-\pi,\pi]$, use an even Fourier series and numerical integration on our calculators to determine the coefficients $a_0, a_1, a_2, a_3, a_4,$ and a_5.

Consider the graphs of $f(x)=e^{-x^2}$ on $[-\pi,\pi]$ with

$F_{E_1}(x)=a_0+a_1\cos(x)$ and $T_1(x)=1-x^2$

$F_{E_2}(x)=a_0+a_1\cos(x)+a_2\cos(2x)$ and $T_2(x)=1-x^2+\dfrac{x^4}{2}$

$F_{E_3}(x)=a_0+a_1\cos(x)+a_2\cos(2x)+a_3\cos(3x)$ and $T_3(x)=1-x^2+\dfrac{x^4}{2}-\dfrac{x^6}{6}$

$F_{E_4}(x)=a_0+a_1\cos(x)+a_2\cos(2x)+a_3\cos(3x)+a_4\cos(4x)$ and $T_4(x)=1-x^2+\dfrac{x^4}{2}-\dfrac{x^6}{6}+\dfrac{x^8}{24}$

and

$F_{E_4}(x)=a_0+a_1\cos(x)+a_2\cos(2x)+a_3\cos(3x)+a_4\cos(4x)+a_5\cos(5x)$ and

$T_5(x)=1-x^2+\dfrac{x^4}{2}-\dfrac{x^6}{6}+\dfrac{x^8}{24}-\dfrac{x^{10}}{120}$

The *T* functions are the corresponding Taylor polynomials for $f(x)=e^{-x^2}$.

To make a least squares comparison of the Taylor polynomial approximation on this interval to the Fourier approximation, compare the values of $\int_{-\pi}^{\pi}\left(f(x)-F_{E_n}(x)\right)^2 dx$ and

$\int_{-\pi}^{\pi}\left(f(x)-T_n(x)\right)^2 dx$ as the number of terms increases.

Which approximation seems to do a better job on of approximating $f(x)=e^{-x^2}$ on the interval $[-\pi,\pi]$?