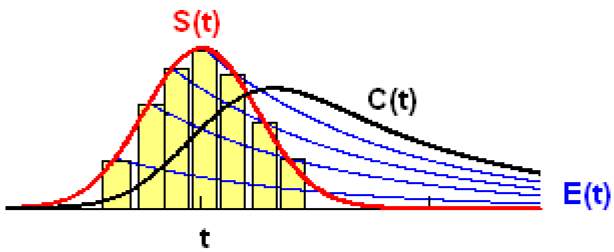
Calculus Challenge #11 **Solution**

In modeling the concentration of hormones in the human body, simple models like the differential equation $\frac{dC}{dt} = -\alpha C(t) + S(t)$ are often used. In this model, $C(t)$ is the concentration of the hormone at time *t* (which is being metabolized at a rate, $\alpha > 0$, proportional to the concentration at that time), while $S(t)$ is the rate at which the hormone is being secreted into the system.

A more sophisticated model for the concentration is

$$C(t) = \int_{-\infty}^{t} S(x)E(t-x)\,dx,$$



where *S* is the rate of secretion and *E* the rate of elimination. For most hormone secretions, $S(t)$ is either a pulse function or a variant of a Gaussian function (shown above).

1) Use technology to sketch a graph of the concentration function $C(t) = \int_{-\infty}^{t} e^{-2x^2} e^{-0.6(t-x)}\,dx$.

The graph above is of the function $C(t) = \int_{-\infty}^{t} e^{-2x^2} e^{-0.6(t-x)}\,dx$ with $S(t) = e^{-2t^2}$.

2) For the integral model, $C(t) = \int_{-\infty}^{t} S(x)E(t-x)\,dx$, find $C(t)$ if $E(t) = e^{-\alpha \cdot t}$ and $S(t) = S$ (constant). Does $\frac{dC}{dt} = -\alpha C(t) + S(t)$ for this model?

If $C(t) = \int_{-\infty}^{t} S(x)E(t-x)\,dx$, and $E(t) = e^{-\alpha \cdot t}$ and $S(t) = S$, then $C(t) = \int_{-\infty}^{t} S \cdot e^{-\alpha \cdot (t-x)}\,dx$, so

$C(t) = Se^{-\alpha \cdot t} \int_{-\infty}^{t} e^{\alpha \cdot x}\,dx = Se^{-\alpha \cdot t}\left(\frac{e^{\alpha \cdot t}}{\alpha}\right) = \frac{S}{\alpha}$. If $C(t) = \frac{S}{\alpha}$ (this is a constant function), then $\frac{dC}{dt} = 0$.

Notice that $-\alpha C(t) + S(t) = -\alpha\left(\frac{S}{\alpha}\right) + S = 0$, so this function does satisfy the original differential equation. Granted, it is not a very interesting solution, but it does work.

3) If $C(t)=\int_{-\infty}^{t} S(x)e^{-\alpha(t-x)}\,dx$, with $S(t)$ a non-constant secretion rate, we want to find $\frac{dC}{dt}$.

To accomplish this, we need to use the definition of derivative, l'Hopital's rule, and the 2nd Fundamental Theorem of Calculus.

a) First, find $C(t+h)$.

If $C(t)=\int_{-\infty}^{t} S(x)e^{-\alpha(t-x)}\,dx$, then $C(t+h)=\int_{-\infty}^{t+h} S(x)e^{-\alpha(t+h-x)}\,dx=e^{-\alpha(t+h)}\int_{-\infty}^{t+h} S(x)e^{\alpha(x)}\,dx$.

So, $C(t+h)=e^{-\alpha(h)}\int_{-\infty}^{t+h} S(x)e^{-\alpha(t-x)}\,dx$.

This can be rewritten as

$$C(t+h)=e^{-\alpha(h)}\left(\int_{-\infty}^{t} S(x)e^{-\alpha(t-x)}\,dx+\int_{t}^{t+h} S(x)e^{-\alpha(t-x)}\,dx\right)=e^{-\alpha(h)}\left(C(t)+\int_{t}^{t+h} S(x)e^{-\alpha(t-x)}\,dx\right)$$

b) Show that

$$C(t+h)-C(t)=\left(e^{-\alpha(h)}C(t)+e^{-\alpha(h)}\int_{t}^{t+h} S(x)e^{-\alpha(t-x)}\,dx\right)-C(t)=$$

$$e^{-\alpha(h)}C(t)-C(t)+e^{-\alpha(h)}\int_{t}^{t+h} S(x)e^{-\alpha(t-x)}\,dx=$$

$$\left(e^{-\alpha h}-1\right)C(t)+e^{-\alpha h}\int_{t}^{t+h} S(x)e^{-\alpha(t-x)}dx.$$

c) Find $\lim_{h\to 0}\frac{C(t+h)-C(t)}{h}$.

Now, we have

$$\lim_{h\to 0}\frac{\left(e^{-\alpha h}-1\right)C(t)+e^{-\alpha h}\int_t^{t+h}S(x)e^{-\alpha(t-x)}dx}{h}=\lim_{h\to 0}\frac{\left(e^{-\alpha h}-1\right)C(t)}{h}+\lim_{h\to 0}\frac{e^{-\alpha h}\int_t^{t+h}S(x)e^{-\alpha(t-x)}dx}{h}.$$

This first limit, $\lim_{h\to 0}\frac{\left(e^{-\alpha h}-1\right)C(t)}{h}$, is of the form 0/0, so we can apply l"Hopital's rule. If $C(t)\cdot\lim_{h\to 0}\frac{\frac{d}{dh}\left(e^{-\alpha h}-1\right)}{\frac{d}{dh}(h)}$ exists, then this limit will be equal to $\lim_{h\to 0}\frac{\left(e^{-\alpha h}-1\right)C(t)}{h}$. So, $C(t)\cdot\lim_{h\to 0}\frac{-\alpha\left(e^{-\alpha h}\right)}{1}=-\alpha\cdot C(t)$ is the limit we want. $\lim_{h\to 0}\frac{\left(e^{-\alpha h}-1\right)C(t)}{h}=-\alpha\cdot C(t)$.

Now, find $\lim_{h\to 0}\frac{e^{-\alpha h}\int_t^{t+h}S(x)e^{-\alpha(t-x)}dx}{h}$. The limit of a product is the product of the limits, provided both exist, so $\left(\lim_{h\to 0}e^{-\alpha h}\right)\left(\lim_{h\to 0}\frac{\int_t^{t+h}S(x)e^{-\alpha(t-x)}dx}{h}\right)=1\cdot\left(\lim_{h\to 0}\frac{\int_t^{t+h}S(x)e^{-\alpha(t-x)}dx}{h}\right)$. It is this final limit we need to find.

So, $\lim_{h\to 0}\frac{\int_t^{t+h}S(x)e^{-\alpha(t-x)}dx}{h}$. This is also an indeterminant form, 0/0, since $\int_t^t S(x)e^{-\alpha(t-x)}dx=0$.

We can again use l'Hopital's rule, only we need to couple it with the second fundamental theorem in the numerator.

We need to compute $\lim_{h \to 0} \frac{\frac{d}{dh}\left(\int_t^{t+h} S(x) e^{-\alpha(t-x)} dx \right)}{\frac{d}{dh}(h)} = \lim_{h \to 0} \frac{\left(S(t+h) e^{-\alpha(t-(t+h))} \right)}{1} = S(t)$. So,

$$\lim_{h \to 0} \frac{\int_t^{t+h} S(x) e^{-\alpha(t-x)} dx}{h} = S(t).$$

$\lim_{h \to 0} \frac{C(t+h) - C(t)}{h} = -\alpha \cdot C(t) + 1 \cdot S(t)$, so $\frac{dC}{dt} = -\alpha \cdot C(t) + S(t)$.

This is just another form of the original differential equation.