Calculus Challenge #13 Solutions due May 20

Skydiving: How fast do we really fall?

Data collected by the United States Parachute Association of a sky diver in free fall is shown in Table 1.

Time (secs)	0	1	2	3	4	5	6	7	8	9	10	11	12
Distance (ft)	0	16	62	138	242	366	504	652	808	971	1138	1309	1483

Table 1: The distance and time for a skydiver in "free fall"

The classical free fall model to describe the location of the skydiver in "free fall" is $x = \frac{1}{2} g t^2$ with $g = 32.2 \text{ ft/sec}^2$. The reason the free fall model is poor that it ignores the resisting force of the air. We are interested in finding a model that includes air resistance which will match the data given in Table 1. Two theories of the effect of air resistance during the free fall have been proposed.

- Theory 1 says that the resistance is proportional to the velocity of the sky diver.
- Theory 2 says that the resistance is proportional to the square of the velocity of the skydiver.

Both theories have been shown to model objects falling through various mediums. Which is correct for a skydiver falling through air? Our goal is to determine which of the two theories gives solutions that match the data.

1. Theory 1 assumes the air resistance is proportional to the velocity of the skydiver, that is,

$$\frac{dv}{dt} = g - \frac{kv}{m}.$$

The differential equation can be written as $\frac{dv}{dt} = \frac{g}{V}(V - v)$ where *V* is the terminal velocity. This revision gives us an equation in terms of parameters we care about. By rewriting the equation in terms of V rather than the arbitrary constant *k/m*, our final solution will be more easily interpreted.

a) Starting with the differential equation $\frac{dv}{dt} = \frac{g}{V}(V - v)$, find the velocity *v* as a function of time *t*. Do you see how the introduction of the parameter *V* helps with interpreting the results of your work? Call your solution to the differential equation with linear resistance $v_1(t)$.

If $\frac{dv}{dt} = g - \frac{kv}{m}$, then $\frac{dv}{dt} = 0$ when $v = \frac{mg}{k} = V$. So, $\frac{k}{m} = \frac{g}{V}$ and $\frac{dv}{dt} = g - \frac{gv}{V} = \frac{g}{V}(V - v)$. This is a variation of the learning curve. By separating variables, we see that $\int \frac{dv}{(V - v)} = \int \frac{g}{V} dt$, so

$-\ln|V-v| = \frac{g}{V}t + c$. Since $V-v>0$, we have $V-v = Ae^{-\frac{g}{V}t}$ with $A>0$. Using the initial condition that $v=0$ when $t=0$, we find that $A=V$. Finally, $v_1(t) = V - Ve^{-\frac{g}{V}t}$.

b) As time passes and the skydiver approaches terminal velocity, the velocity should approach *V*. Does your function $v_1(t)$ indicate that this is true?

Since $v_1(t) = V - Ve^{-\frac{g}{V}t}$, we see that *v* approaches terminal velocity V exponentially from below.

c) Use $\frac{dx_1}{dt} = v_1(t)$ to solve for distance as a function of time.. i) As time passes, the distance should be increasing linearly. Does this models describe a linear growth in distance as $t \to \infty$?

For Theory 1, we have $v_1(t) = V - Ve^{-\frac{g}{V}t}$, so $x_1(t) = \int V - Ve^{-\frac{g}{V}t}dt = Vt + \left(\frac{V^2}{g}\right)e^{-\frac{g}{V}t} + C$ since the initial velocity was 0. The function $x_1(t) = Vt + \left(\frac{V^2}{g}\right)e^{-\frac{g}{V}t} - \frac{V^2}{g}$ reduces to a linear equation $x_1(t) = Vt - \left(\frac{V^2}{g}\right)$ as $t \to \infty$.

2. Theory 2 assumes the air resistance is proportional to the square of the velocity of the skydiver, that is,

$$\frac{dv}{dt} = g - \frac{kv^2}{m}.$$

a) Rewrite this differential equation in terms of the terminal velocity V.

If $\frac{dv}{dt} = 0$, then $g = \frac{kV^2}{m}$, so $\frac{k}{m} = \frac{g}{V^2}$. We have $\frac{dv}{dt} = g - \left(\frac{g}{V^2}\right)v^2 = \left(\frac{g}{V^2}\right)\left(V^2 - v^2\right)$. We need to solve the differential equation $\frac{dv}{dt} = \left(\frac{g}{V^2}\right)\left(V^2 - v^2\right)$.

b) Show that under these conditions, $v_2(t)=V\left(\frac{e^{\frac{2g}{V}t}-1}{e^{\frac{2g}{V}t}+1}\right)$. Does the skydiver approach terminal velocity as time passes?

If $\frac{dv}{dt}=\left(\frac{g}{V^2}\right)\left(V^2-v^2\right)$, we separate variables to find $\int\frac{dv}{\left(V^2-v^2\right)}=\int\left(\frac{g}{V^2}\right)dt$. This can be done using partial fractions. The right side is just $\int\left(\frac{g}{V^2}\right)dt=\frac{g\,t}{V^2}+c$ and the left side can be solved as well. $\int\frac{dv}{\left(V^2-v^2\right)}=\int\frac{\left(\frac{1}{2V}\right)}{V+v}+\frac{\left(\frac{1}{2V}\right)}{V-v}dv=\left(\frac{1}{2V}\right)\left(\ln\left(V+v\right)-\ln\left(V-v\right)\right)=\left(\frac{1}{2V}\right)\left(\ln\left(\frac{V+v}{V-v}\right)\right)$.

So $\ln\left(\frac{V+v}{V-v}\right)=\frac{2g\,t}{V}+c$ and $\left(\frac{V+v}{V-v}\right)=Ce^{\frac{2gt}{V}}$. If t = 0, v = 0, so $1=C$ and $\left(\frac{V+v}{V-v}\right)=e^{\frac{2gt}{V}}$.

Solving for v, we have $V+v=Ve^{\frac{2gt}{V}}-ve^{\frac{2gt}{V}}$ and $v\left(1+e^{\frac{2gt}{V}}\right)=Ve^{\frac{2gt}{V}}-V$. Finally, we have $v_2(t)=V\left(\frac{e^{\frac{2gt}{V}}-1}{e^{\frac{2gt}{V}}+1}\right)$. We see that $\lim_{t\to\infty}\left(\frac{e^{\frac{2gt}{V}}-1}{e^{\frac{2gt}{V}}+1}\right)=1$, so we approach terminal velocity as expected.

c) Use $\frac{dx_2}{dt}=v_2(t)$ to solve for distance as a function of time. As $t\to\infty$, the distance should be increasing linearly. Does this model describe a linear growth in distance?

For Theory 2, we have $v_2(t)=V\left(\frac{e^{\frac{2gt}{V}}-1}{e^{\frac{2gt}{V}}+1}\right)$, so $x_2(t)=\int V\left(\frac{e^{\frac{2gt}{V}}-1}{e^{\frac{2gt}{V}}+1}\right)dt$. “All we have to do” is find this antiderivative. Let $c=\frac{2g}{V}$ to keep the work somewhat easier to read. We need to integrate $\int\left(\frac{e^{ct}-1}{e^{ct}+1}\right)dt$. For this we need to be a little bit clever. We need a *u*-substitution. Let $u=e^{ct}$, so $du=ce^{ct}dt$ and $\frac{du}{ce^{ct}}=dt$. Then $\int\left(\frac{e^{ct}-1}{e^{ct}+1}\right)dt=\frac{1}{c}\int\frac{u-1}{u\left(u+1\right)}du$ which we can do by partial fractions.

$$\frac{A}{u}+\frac{B}{u+1}=\frac{(A+B)u+A}{u(u+1)} \text{ so } A=-1 \text{ and } B=2\text{, so}$$

$$\frac{1}{c}\int\frac{u-1}{u(u+1)}du=\frac{1}{c}\int\frac{2}{(u+1)}-\frac{1}{u}du=\left(\frac{1}{c}\right)2\ln(u+1)-\left(\frac{1}{c}\right)\ln(u)+C.$$

Putting it all together, we have $x_2(t)=\int V\left(\frac{e^{\frac{2gt}{V}}-1}{e^{\frac{2gt}{V}}+1}\right)dt=\left(\frac{V}{c}\right)\ln\left(\frac{(u+1)^2}{u}\right)+C$ with $u=e^{ct}$ and $c=\frac{2g}{V}$, so $x_2(t)=\left(\frac{V^2}{g}\right)\ln\left(e^{\frac{2g}{V}t}+1\right)-Vt+C$. Since $x=0$ when $t=0$, we find $C=-\left(\frac{V^2}{2g}\right)\ln(2)$.

So, finally, we have $x_2(t)=\left(\frac{V^2}{g}\right)\ln\left(e^{\frac{2g}{V}t}+1\right)-Vt-\frac{V^2\ln(2)}{2g}$.

Does this approach as linear function as $t\to\infty$? We see that $\lim_{t\to\infty}\ln\left(e^{\frac{2g}{V}t}+1\right)=\frac{2gt}{V}$, so we approach $x_2(t)\to\left(\frac{V^2}{g}\right)\left(\frac{2gt}{V}\right)-Vt-\frac{V^2\ln(2)}{2g}=Vt-\frac{V^2\ln(2)}{2g}$, which is a linear function with slope *V*.

3. Use the data to estimate the terminal velocity V. With this value of *V*, graph the two distance functions ($x_1(t)$ and $x_2(t)$) against the data. Which best describes the values given by the United States Parachute Association?

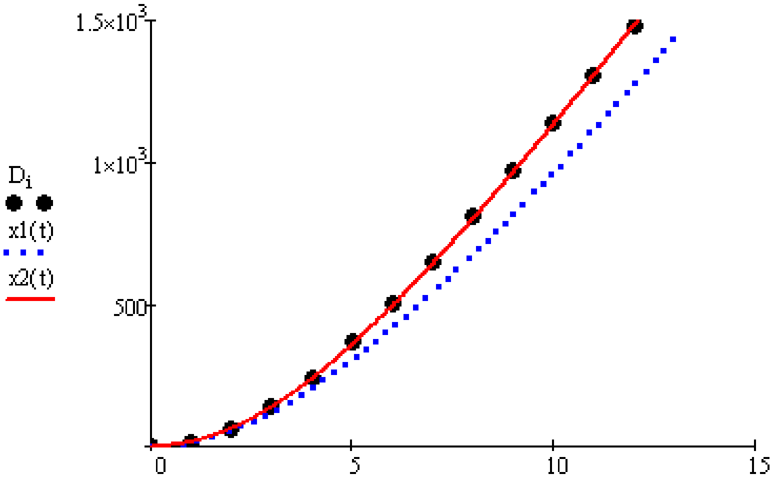
Time (secs)	0	1	2	3	4	5	6	7	8	9	10	11	12
Distance (ft)	0	16	62	138	242	366	504	652	808	971	1138	1309	1483
Velocity (ft/sec)		32	61	90	114	131	143	152	160	165	169	172	

Table 1: Using symmetric differences to approximate velocity.

The pattern of differences suggests something around 180 would be about right (many other values could be chosen). So, we have the contenders.

$x_1(t)=Vt+\left(\frac{V^2}{g}\right)e^{-\frac{g}{V}t}-\frac{V^2}{g}$ and $x_2(t)=\left(\frac{V^2}{g}\right)\ln\left(e^{\frac{2g}{V}t}+1\right)-Vt-\frac{V^2\ln(2)}{2g}$ with $g=32.2$ and $V=180$.

$$x1(t) := V\cdot t + \frac{V^2}{g}\cdot e^{\frac{-g}{V}\cdot t} - \frac{V^2}{g} \qquad x2(t) := \frac{V^2}{g}\cdot \ln\left(e^{\frac{2\cdot g}{V}\cdot t} + 1\right) - V\cdot t - \frac{V^2}{2\cdot g}\cdot \ln(2)$$



We see that Model 2, with quadratic air resistance is the most appropriate model.