

1. Before we begin, think about the function $f(x) = \tan^{-1}\left(\frac{\sqrt{x} + \sqrt{k}}{1 - \sqrt{kx}}\right)$ from a precalculus

point of view. The function f has a domain of $\{x \in \mathfrak{R} \mid x \neq \frac{1}{k}\}$, since $x = \frac{1}{k}$ would give us a zero denominator. The inverse tangent function is an odd function, which means that its value is positive when the input from the domain is positive and negative, when the input is negative. Since the denominator changes signs at $x = \frac{1}{k}$ and the numerator is always positive, we should

expect some fireworks of some kind at $x = \frac{1}{k}$. Moreover, $\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2}$ and

$\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$, we should expect a jump between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Now, add to that what we learn from calculus.

If $f(x) = \tan^{-1}\left(\frac{\sqrt{x} + \sqrt{k}}{1 - \sqrt{kx}}\right)$, $x \neq \frac{1}{k}$, then

$$f'(x) = \frac{1}{1 + \left(\frac{\sqrt{x} + \sqrt{k}}{1 - \sqrt{kx}}\right)^2} \cdot \left(\frac{(1 - \sqrt{kx})\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x} + \sqrt{k})\left(\frac{\sqrt{k}}{2\sqrt{x}}\right)}{(1 - \sqrt{kx})^2} \right), x \neq \frac{1}{k}.$$

Simplifying this expression, we have

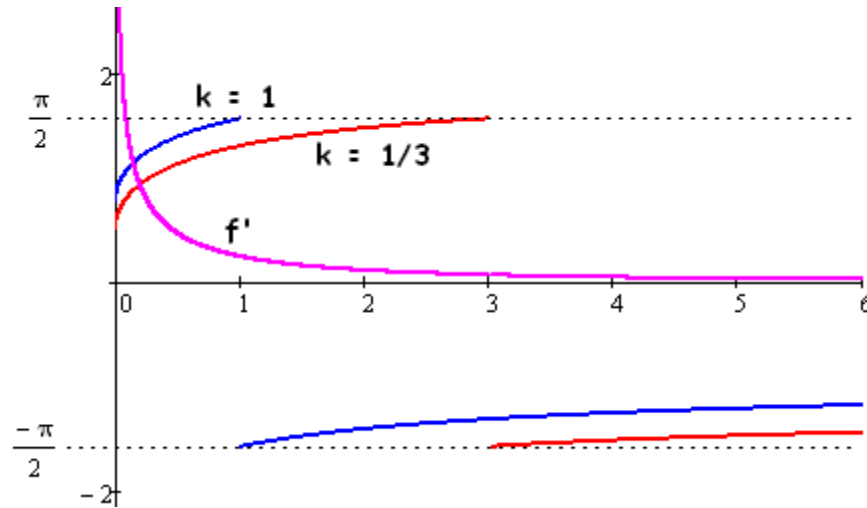
$$f'(x) = \frac{\left(\frac{1+k}{2\sqrt{x}}\right)}{(1 - \sqrt{kx})^2 + (\sqrt{x} + \sqrt{k})^2} = \frac{1+k}{2\sqrt{x}(1+x+k+kx)} = \frac{1}{2\sqrt{x}(1+x)}, \quad x \neq \frac{1}{k}$$

Notice that the value of the derivative does not depend upon k . This is similar to $y = \ln(kx)$, with $y' = \frac{1}{x}$. The value of the derivative does not depend upon k , but the graph of the function does depend upon k . The consequence is the curves differ by only a vertical shift.

b) We have $f'(x) = \frac{1}{2\sqrt{x}(1+x)}$, $x \neq \frac{1}{k}$, so $f'(k) = \frac{1}{2\sqrt{k}(1+k)}$, but, $f'\left(\frac{1}{k}\right)$ does not exist, since the function cannot be continuous there. There is a jump of π units there.

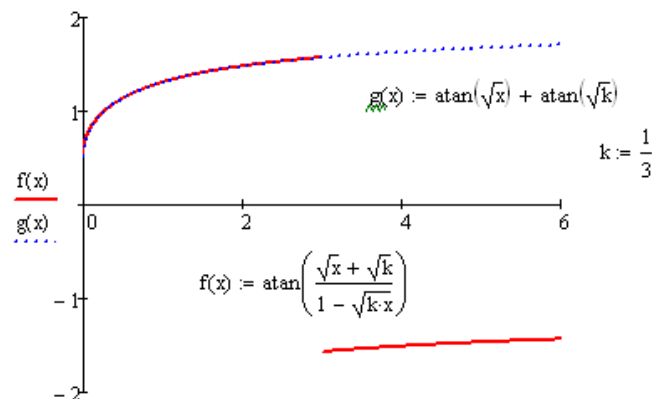
c) Below, we see the graph $f(x) = \tan^{-1}\left(\frac{\sqrt{x} + \sqrt{k}}{1 - \sqrt{kx}}\right)$ for $k = 1$ and $k = \frac{1}{3}$ along with the

derivative. Since the derivatives are equal everywhere they are defined, the graphs are vertical shifts of each other (only the size of the shift changes abruptly).



Students made two mistakes on this problem, the first was to equate our given function, $f(x) = \tan^{-1}\left(\frac{\sqrt{x} + \sqrt{k}}{1 - \sqrt{kx}}\right)$, to $g(x) = \tan^{-1}(\sqrt{x}) + \tan^{-1}(\sqrt{k})$, because $\tan(f(x)) = \tan(g(x))$.

This was a clever use of trig identities, but unfortunately is incorrect. Consider $\tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right)$, but that doesn't mean that $\left(\frac{\pi}{4}\right) = \left(\frac{3\pi}{4}\right)$. Comparing graphs would also illustrate the problem.



However, the derivatives of f and g ARE equal. Again we see that if two functions have equal derivatives, that means one is a vertical shift of the other.

The second error was in not factoring the denominator in $f'(x) = \frac{1+k}{2\sqrt{x}(1+x+k+kx)}$

$$\text{into } f'(x) = \frac{1+k}{2\sqrt{x}((1+k)+x(1+k))} = \frac{\cancel{(1+k)}}{2\sqrt{x}\cancel{(1+k)}(1+x)}.$$

By failing to factor out $1+k$, the result appeared to depend upon k , when in fact, it doesn't.

But even with those mistakes, the kids did some very nice work with some messy derivatives and integrals.

2. This part of the challenge requires students to play around with the basic rules for adding integrals.

$$\text{Let } H(x) = f(x) + c \text{ with } \int_a^b f(x) dx = \int_b^c H(x) dx \text{ and } \int_b^c f(x) dx = abc.$$

a) Find $\int_a^c f(x) dx$. Here we know that $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$ and

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_b^c H(x) dx + abc. \text{ Since } H(x) = f(x) + c,$$

$$\int_b^c H(x) dx = \int_b^c f(x) + c dx = abc + c(c-b). \text{ Finally, } \int_a^c f(x) dx = 2abc + c(c-b)$$

b) Find the average value of H on $[a, c]$. This is

$$\frac{\int_a^c H(x) dx}{c-a} = \frac{\int_a^c f(x) + c dx}{c-a} = \frac{2abc + c(c-b) + c(c-a)}{c-a} = \frac{2abc + 2c^2 - ca - cb}{c-a}.$$