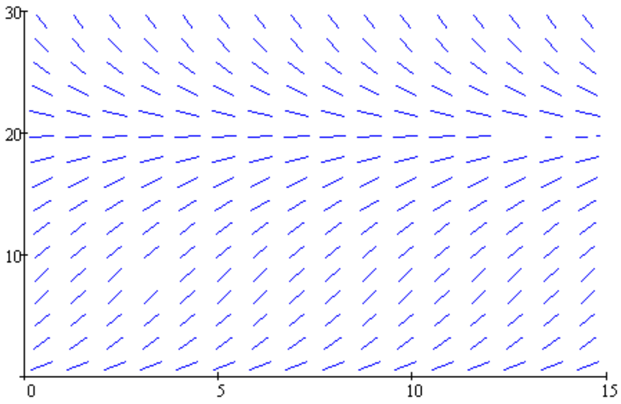
Calculus Challenge #8

SOLUTION

Observations on the growth of animal tumors indicate the size *y* of the tumor obeys the differential equation $\frac{dy}{dt} = -ky\ln\left(\frac{y}{b}\right)$ with *k* and *b* positive constants. This is known as the Gompertz growth law.

It is always a good first step to look at the slope field. At right we see the slope field for the differential equation using specific choices of *k* and *b*. All choices for *k* and *b* produce similar graphs.

This graph should help guide your investigation.



Slope field for dy/dt = -k y ln(y/b)

When working with a differential equation, we should always consider the questions of domain and range and stable points before doing any messy algebra. In this case, we know that $y > 0$ and $\frac{dy}{dt} = 0$ when $y = b$.

a) In terms of *k* and *b*, when is the tumor growing most rapidly?

The tumor is growing most rapidly when $\frac{dy}{dt}$ is at a maximum. So, we perform the classic optimization process on $\frac{dy}{dt}$ by setting the derivative of $\frac{dy}{dt}$ equal to zero. So,

$$\frac{d^2y}{dt^2} = \frac{-ky\left(\frac{dy}{dt}\right)}{b\left(\frac{y}{b}\right)} - k\left(\frac{dy}{dt}\right)\ln\left(\frac{y}{b}\right) = -k\left(\frac{dy}{dt}\right)\left(1+\ln\left(\frac{y}{b}\right)\right) = 0.$$

Either $\frac{dy}{dt} = -ky\ln\left(\frac{y}{b}\right) = 0$ or $1+\ln\left(\frac{y}{b}\right) = 0$. In the first case, $y = b$ is a stable solution (as we already noted) and in the second case, $\ln\left(\frac{y}{b}\right) = -1$ so $y = \frac{b}{e}$. If *y* is not constant, then $y = \frac{b}{e}$ is the size of the tumor when the growth rate is a maximum. The maximum growth rate is $\frac{dy}{dt}\left(\frac{b}{e}\right) = -k\left(\frac{b}{e}\right)\ln\left(\frac{\left(\frac{b}{e}\right)}{b}\right) = \frac{kb}{e}$. Notice also that if $y > b$, then the function is strictly decreasing (we will use this fact later) and there is no maximum value.

We are asked *when* is the tumor growing most rapidly, so we need to find the value of *t* associated with $y = {}^{b}\!/_{e}$. To find *t*, we need to solve the equation, which is problem b).

b) Solve the differential equation with the initial condition $y(0) = y_0$.

If $\dfrac{dy}{dt} = -ky\ln\left({}^{y}\!/_{b}\right)$, then $\displaystyle\int \frac{dy}{y\ln\left({}^{y}\!/_{b}\right)} = \int -kdt + C$. The left side is a *u*-substitution with $u = \ln\left({}^{y}\!/_{b}\right)$. So, $\ln\left|\ln\left({}^{y}\!/_{b}\right)\right| = -kt + C$ and $\left|\ln\left({}^{y}\!/_{b}\right)\right| = Ae^{-kt}$, where $A = e^{C}$ and is therefore positive. We always have to consider the absolute value issue.

If $y < b$, then $\ln\left({}^{y}\!/_{b}\right) < 0$ and $\left|\ln\left({}^{y}\!/_{b}\right)\right| = -\ln\left({}^{y}\!/_{b}\right)$. This gives us $\ln\left({}^{y}\!/_{b}\right) = -Ae^{-kt}$ Exponentiating once again and solving for *y* gives

$$y(t) = be^{-Ae^{-kt}}.$$

If $y(0) = y_0$, then $y_0 = be^{-A}$ which requires $A = -\ln\left({}^{y_0}\!/_{b}\right)$.

$$\text{So, } y(t) = be^{\ln\left(\frac{y_0}{b}\right)e^{-kt}} \quad \text{when } y < b.$$

If *y* > *b*, then $y(t) = be^{\ln\left(\frac{b}{y_0}\right)e^{-kt}}$.

To complete part a) we want the *t* for which $y(t) = \dfrac{b}{e}$, we must solve the equation

$$\frac{b}{e} = be^{\ln\left({}^{y_0}\!/_{b}\right)e^{-kt}}.$$

This is a mess, but by persevering, we find $\dfrac{-1}{\ln\left({}^{y_0}\!/_{b}\right)} = e^{-kt}$ and $t = -\left(\dfrac{1}{k}\right)\ln\left(\dfrac{-1}{\ln\left({}^{y_0}\!/_{b}\right)}\right)$.

So, now we have answered the first question.

c) Find a relation between y_0, *k*, and *b* so that the graph of *y* vs *t* has no point of inflection.

As we can see from the slope field, there is no point of inflection, then $y \geq b$. From $\dfrac{d^2y}{dt^2} = -k\left({}^{dy}\!/_{dt}\right)\left(1 + \ln\left({}^{y}\!/_{b}\right)\right)$, we see that there is no zero for the second derivative if $y > b$. If $y = b$, then $y_0 = b$. As a real-world problem, our equation says that $y > b$ is not possible.

d) Find $\lim_{t\to\infty} y(t)$. What does this say about the growth of the tumor?

The limit $\lim_{t\to\infty} be^{\ln\left(b/y_0\right)e^{-kt}} = be^0 = b$. The parameter b represents the maximum size of the tumor. Once it reaches this size, it cannot grow any more.

e) Given the slope field graphed above, what values were used for k and b?

You can't know for sure, of course, but we can get some estimates. Since $\frac{dy}{dt} = 0$ when $y = b$, from the slope field it looks like b = 20 (or thereabout).

From our earlier work, we found the maximum growth rate to be $\frac{kb}{e}$ at $\left(-\left(\frac{1}{k}\right)\ln\left(\frac{-1}{\ln\left(y_0/b\right)}\right), \frac{b}{e}\right)$. If b = 20, this is a slope of about 7.4k at y = 7.4. So, putting the graph in Paint, I captured the slope segment at y = 7.4, extended it from 0 to 10, and estimate the slope as 3.1. This should be 7.4k, so, $k \approx 0.4$. Other approaches may work better, but that is what I thought to do.