Calculus Challenge #9 SOLUTION

1. A bullet passes through a piece of wood 10 cm thick. It enters the wood at 200 m/sec and exits at 80 m/sec. It is assumed that the velocity *v* of the bullet while in the wood obeys the differential equation $\frac{dv}{dt} = -kv^2$.

a) How long did it take the bullet to pass through the wood?

b) Do you think this model is realistic? Explain why or why not.

(Both Problems are modified from Lomen and Lovelock, *Exploring Differential Equations via Graphics and Data*, *Preliminary Edition*, John Wiley and Sons, 1996)

Solution: a) We are given $x_0 = 0, v_0 = 200$ and $x_{t^*} = 10, v_{t^*} = 80$ for some time t^*. If $\frac{dv}{dt} = -kv^2$, then $\int \frac{dv}{v^2} = \int -kdt + c$ and $\frac{-1}{v} = -kt + c$. Solving for *v*, we find $v(t) = \frac{1}{kt + c}$. Using the initial conditions, we have $200 = \frac{1}{0 + c}$ or $c = 0.005$. So, the solution is $v(t) = \frac{1}{kt + 0.005}$. Integrating again, we find $x(t) = \left(\frac{1}{k}\right)\ln(kt + 0.005) + c$ with $c = -\frac{1}{k}\ln(0.005)$.

So, $v(t) = \frac{1}{kt + 0.005}$ and $x(t) = \left(\frac{1}{k}\right)\left(\ln(kt + 0.005) - \ln(0.005)\right)$.

If t^* is the time at which the bullet has passed through the board, then $80 = \frac{1}{kt^* + 0.005}$ and $0.1 = \left(\frac{1}{k}\right)\left(\ln(kt^* + 0.005) - \ln(0.005)\right)$.

From the first equation we have $kt^* + 0.005 = \frac{1}{80} = 0.0125$, so substituting for $kt^* + 0.005$ in the second equation and solving for *k* we have

$$k = (10)\left(\ln(0.0125) - \ln(0.005)\right) = 10\ln(5/2) \approx 9.1629.$$

Our solution is $v(t) = \frac{1}{9.1629t + 0.005} = \frac{200}{2000\ln(5/2)t + 1}$ and $x(t) = \frac{\ln\left(2000\ln(5/2)t + 1\right)}{10\ln(5/2)}$.

So, $0.1 = \frac{\ln\left(2000\ln(5/2)t + 1\right)}{10\ln(5/2)}$ gives $\frac{(5/2) - 1}{2000\ln(5/2)} = t^*$ or $t^* \approx 0.0008185$ seconds.

b) Do you think this model is realistic? Explain why or why not.

No, $v(t) = \dfrac{200}{2000\ln\left(\frac{5}{2}\right)t + 1}$ is not a realistic model in general since the velocity can never be zero. If $v \neq 0$, then there is no board thick enough to stop a bullet, which we know is not true. It may describe the deceleration of a bullet in thin boards, but cannot model the process in general.

2. Richard Zimmer, while working on a Science Fair project as a student at Canyon del Oro High School in Tucson, Arizona, shot a 120 grain hollow point bullet from a 300 Winchester magnum rifle and measured the velocity of the bullet as a function of distance traveled.

Distance (ft)	0	300	600	900	1200	1500
Velocity (ft/sec)	3290	2951	2636	2342	2068	1813

It is claimed that a bullet, when fired horizontally, is subject only to air resistance which is proportional to some power of velocity. This means that $\dfrac{dv}{dt} = -k\,v^n$.

a) Find $\dfrac{dv}{dx}$.

b) Use the difference quotients from the data to estimate the values of *k* and *n*.

c) Find the velocity (*v*) as a function of distance (*x*).

(Both Problems are modified from Lomen and Lovelock, *Exploring Differential Equations via Graphics and Data*, *Preliminary Edition*, John Wiley and Sons, 1996)

a) By the chain rule, we know that $\dfrac{dv}{dt} = \dfrac{dv}{dx}\cdot\dfrac{dx}{dt}$ and $\dfrac{dx}{dt} = v$, so $\dfrac{dv}{dt} = v\cdot\dfrac{dv}{dx}$.

If $\dfrac{dv}{dt} = -kv^n$, then $v\dfrac{dv}{dx} = -kv^n$ and $\dfrac{dv}{dx} = -kv^{n-1}$.

b) We can use difference quotients to approximate $\frac{dv}{dx}$ and its associated velocity v. We associate the slope between adjacent points with the midpoint velocity.

Slope -1.13 -1.05 -0.98 -0.913 -0.850

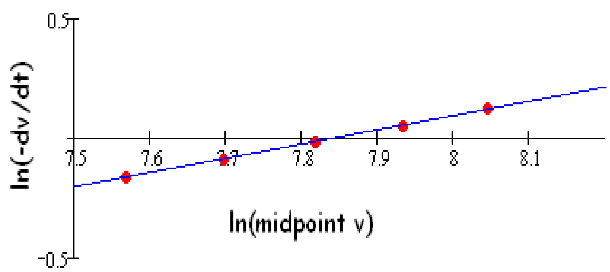
Distance (ft)	0	300	600	900	1200	1500
Velocity (ft/sec)	3290	2951	2636	2342	2068	1813

Average velocity 3120.5 2793.5 2489 2205 1940.5

This generates the table below:

Dv/dx	-1.13	-1.05	-0.98	-0.913	-0.850
Velocity	3120.5	2793.5	2489	2205	1940.5

Using the data above, if $\frac{dv}{dx} = -kv^{n-1}$, then $\ln\left(-\frac{dv}{dx}\right) = \ln k + (n-1)\ln v$. We had to multiply by -1 since $\frac{dv}{dx} < 0$. If we fit a line to the data, we see that



$$\ln\left(-\frac{dv}{dx}\right) = -4.68697 + .59729\ln v.$$

So, exponentiating both sides we see that $\frac{dv}{dx} = -.0092145v^{.59719}$.

We find that $k = .0092145$ and $n = 1.59729$.

c) All that is left is to solve this differential equation by separation of variables.

If $\frac{dv}{dx} = -.0092145v^{.59719}$, then $\int v^{-.59719}dv = \int -.0092145dx + c$

and

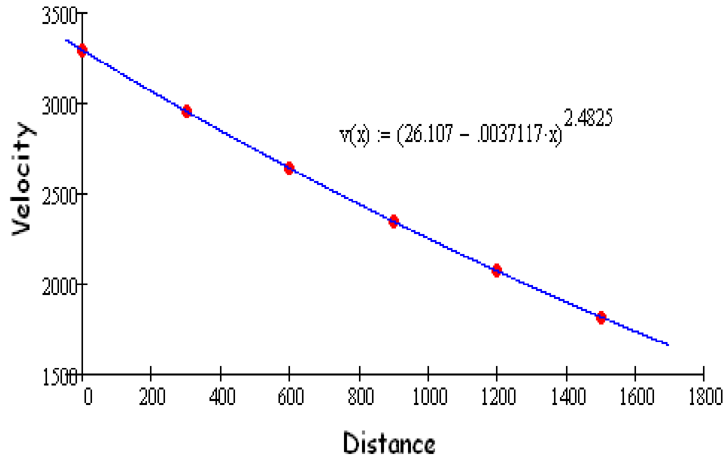
$$\frac{v^{.40281}}{.40281} = -.0092145x + c \quad \text{or} \quad v^{.40281} = -.0037117x + 26.107,$$

using the initial condition of (0, 3290).

Finally,

$$v = \left(26.107 - .0037117x\right)^{2.4825},$$

which fits the data well as shown in the figure below.



Other formulations also work well, so it is a good idea to check your result by graphing it against the original data.