Calculus Challenge Problem #3

Due by the end of the day on Wednesday, October 29

One important theorem at this point in the course is known as the product rule.

If f and g are differentiable functions, then $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + f'(x)\cdot g(x).$

In this challenge, we want you to extend and generalize the product rule. The results may be simpler to read if we suppress the function notation and just write $\frac{d}{dx}\left(f\cdot g\right) = f\cdot g' + f'\cdot g$ with the understanding that each function is a function of x. It is assumed that all functions are at least n times differentiable.

1. Extend the product rule to more than two differentiable functions.

a) $\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\cdot f_3(x)\right)$

b) $\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\cdot f_3(x)\cdot f_4(x)\right)$

c) Find the derivative for a product of k differentiable functions $\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\cdots f_k(x)\right)$

2. Find a product rule for:

   a) the second derivative $\frac{d^2}{dx^2}\left(f(x)\cdot g(x)\right)$.

   b) the third derivative $\frac{d^3}{dx^3}\left(f(x)\cdot g(x)\right)$.

   c) Look at the 4th and 5th derivatives to generalize to the n^{th} derivative $\frac{d^n}{dx^n}\left(f(x)\cdot g(x)\right)$. Explain the pattern that you see in the coefficients.

3. Combine these ideas to find $\frac{d^2}{dx^2}\left(f\cdot g\cdot h\right)$ and $\frac{d^3}{dx^3}\left(f\cdot g\cdot h\right)$. Explain the pattern you see in the coefficients.

4. Suppose I find $\frac{d^5}{dx^5}\left(f\cdot g\cdot h\cdot k\right)$. Is there a $f''g\,h'k''$ term in the expansion? If there is, what is its coefficient? If there isn't, explain why this term does not exist.

(try to answer this question without actually computing the 5th derivative)