Calculus Challenge Problem #6 **Due January 14, 2009**

At this time of year, students in many schools are beginning to prepare for mid-term or final exams. Those in schools that have exams before the winter break will not have time to work on this problem right now. Those schools that have exams after the break will have time, but can't participate at the beginning of January.

So, this problem is due on January 14, 2009.

If you have exams now, save the problem until after the break. If you have exams later, do it now. This problem is a little shorter than the others as well (to help you fit it in to your schedule) but it is a lot more intensively algebraic, as you will soon see.

The normal distribution has the probability density function

$$N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \text{, with } -\infty < x < \infty .$$

The *t*-distribution has the probability density function

$$t(x) = \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} \text{, with } -\infty < x < \infty$$

and d representing an integer constant known as the degrees of freedom.

(If you are wondering about the two factorials $\left(\frac{d-1}{2}\right)!$ and $\left(\frac{d-2}{2}\right)!$, yes there is such a thing as a half-factorial. Type 4.5! into your TI-84 and see what it gives you. For an explanation of these half-factorials, see *Calculus and Factorials*. The derivation uses integrals which you probably have not seen yet, but that's not important for this problem.)

One of the fundamental theorems of statistics is that the *t*-distribution approaches the normal distribution as the degrees of freedom increase without bound. That is,

$$\lim_{d\to\infty} t(x) = N(x).$$

And that's what you are going to prove with the help of L'Hopital's Rule, Stirling's approximation for factorials and your algebra skills.

First, we will break the problem into two pieces and use the theorem that the limit of a product is the product of the limits, provided the two limits exist.

$$t(x) = \left(\frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!} \right) \left(\frac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} \right)$$

. We will show that the constant in front has a limit of $\frac{1}{\sqrt{2\pi}}$ and the function in back has a limit of $e^{-\frac{1}{2}x^2}$.

1) Use L'Hopital's Rule to show that $\lim_{d \to \infty} \left(\frac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} \right) = e^{-\frac{1}{2}x^2}$.

2) Stirling's approximation for factorials is $n! \approx \sqrt{2\pi n}\, e^{-n} n^n$. See *Stirling's Formula* for a derivation of Stirling's formula. (As with the half-factorials, the derivation isn't important for this problem, it's just background information).

Approximate the factorials with $n! \approx \sqrt{2\pi n}\, e^{-n} n^n$, factor and group carefully, and use L'Hopital's Rule when necessary to show that

$$\lim_{d \to \infty} \left(\frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!} \right) = \frac{1}{\sqrt{2\pi}} .$$

3. Put 1) and 2) together to show that $\lim_{d \to \infty} t(x) = N(x)$ and show your AP Stat teacher your result. She will be pleased.