Calculus and Factorials

Calculus students are familiar with the factorial and know $n!$ as the sum of the first n in positive integers, that is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. We can also define factorials recursively with $0! = 1$ and $n! = n\cdot(n-1)!$ to achieve the same result. A more formal representation is

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n\cdot f(n-1) & \text{if } n > 0 \quad \text{and } n \in \mathbb{Z}^{+}. \end{cases}$$

This definition is often the student's first experience with recursive definitions. Students often wonder why 0! = 1, but are not surprised when they type 8! into their TI-83 the value 40320 shows on the screen. They are always quite surprised to see 119292.462 on the screen when they type in 8.5!. Why would anyone want to know 8.5!? One reason is that the probability density function for what is known as the *t*-distribution is

$$P(x) = \frac{1}{\sqrt{n\pi}}\left(\frac{\frac{n-1}{2}!}{\frac{n-2}{2}!}\right)\left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}$$

where n is a positive integer and is called the degrees of freedom. The important part of this is that both $\left(\frac{n}{2}\right)!$ and $\left(\frac{n-1}{2}\right)!$ are terms in the function. Both cannot be integers! Any calculator able to handle *t*-tests must be able to compute numbers like 8.5!. To understand what 8.5! is and how it is computed requires calculus. Fortunately for us, we have been studying calculus all year, so we should be able to figure this out.

Factorials Revisited

Consider the improper integral $\int_0^\infty x^n e^{-x}\,dx$. The value of the integral depends on the value of n. So we can define a function

$$F(n) = \int_0^\infty x^n e^{-x}\,dx.$$

To evaluate the integral, use integration by parts. So

$$\begin{aligned} u &= x^n & dv &= e^{-x}dx \\ du &= n\,x^{n-1}dx & v &= -e^{-x} \end{aligned}$$

and

$$\int_0^\infty x^n e^{-x}\,dx = \lim_{k\to\infty}\left(-x^n e^{-x}\right)\Big|_0^k + n\int_0^\infty x^{n-1}e^{-x}\,dx = 0 + n\int_0^\infty x^{n-1}e^{-x}\,dx$$

This means that $F(n) = nF(n-1)$. That's the basic definition of the factorial. If we can also show that $F(0) = 1$, then the function defined by our integral will indeed be the factorial function we know and love and we can use it to determine *n*! when *n* is not an integer.

Is $F(0) = \int_0^\infty x^0 e^{-x}\,dx = 1$? The integral $\int_0^\infty e^{-x}\,dx$ is a standard improper integral and students can easily show that $\int_0^\infty e^{-x}\,dx = 1$. The function $F(n) = \int_0^\infty x^n e^{-x}\,dx$ <u>is</u> the factorial function for all positive integers *n*. (This also gives another explanation for why 0! = 1.) Use your calculator to evaluate $\int_0^\infty x^n e^{-x}\,dx$ for $n = 3,\ 4, 5, 6, \ldots$. You can use 100 for the upper limit of integration. This factorial function is a variation of the the gamma function $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$ which has important applications in statistics.

So what about 8.5!? Using the numerical integration feature of your calculator to evaluate $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0.5, 1.5,\ 2.5,\ 3.5, \ldots$. You should notice that 3.5! fits nicely between 3! and 4!. You should also notice that your calculator can evaluate 8.5! much more quickly than $\int_0^\infty x^{8.5} e^{-x}\,dx$, so it must have some other way to do this.

When we found that $\int_0^\infty x^n e^{-x}\,dx = n\int_0^\infty x^{n-1} e^{-x}\,dx$, or $F(n) = nF(n-1)$, we did not require *n* to be an integer. This result is true for any real number *n*. We only required *n* to be a non-negative integer to match the value with our old friend *n*!. So it is still true that 8.5! = $(8.5)7.5! = (8.5)(7.5)6.5! = (8.5)(7.5)(6.5)(5.5)(4.5)(3.5)(2.5)(1.5).5!$. So, all that is required is to find a new initial value. This time instead of 0!, we need 0.5!. This integral is a little more difficult than those done earlier.

To evaluate $\int_0^\infty \sqrt{x} e^{-x}\,dx$ we make the substitution $\sqrt{x} = y$, so $x = y^2$ and $dx = 2y\,dy$.

With this substitution, we have $\int_0^\infty \sqrt{x} e^{-x}\,dx = \int_0^\infty 2y^2 e^{-y^2}\,dy$, which can be evaluated using parts.

Let

$$u = y \qquad dv = 2ye^{-y^2}dx$$

$$du = dy \qquad v = -e^{-y^2}$$

$$\text{so } \int_0^\infty 2y^2 e^{-y^2}\,dy = \lim_{k\to\infty}\left(-ye^{-y^2}\Big|_0^k\right) + \int_0^\infty e^{-y^2}\,dy = 0 + \int_0^\infty e^{-y^2}\,dy\,.$$

This last integral is well known, with $\int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}$, but its evaluation is beyond the first course in calculus. (See http://courses.ncssm.edu/math/TALKS/PDFS/normal.pdf)

So, $8.5! = (8.5)(7.5)(6.5)(5.5)(4.5)(3.5)(2.5)(1.5)\frac{\sqrt{\pi}}{2}$, and the “half-factorials” can be computed with the recursive formula

$$f(n) = \begin{cases} \frac{\sqrt{\pi}}{2} & \text{if } n = \frac{1}{2} \\ n \cdot f(n-1) & \text{if } n > \frac{1}{2} \end{cases},$$

with $n = k + 0.5$ for all non-negative integers *k*.

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