Calculus Challenge #1

Solution:

Consider the situation where player X has hit 20 out of 26 shots, so $\frac{20}{26} = 0.7695$. The announcer would never say, "Player X is making 76.95% of his free throws." He would round this off and say the he was making 77% of his free throws. The numbers 78% and 76% are both rounded values. So, if we let *x* represent the number of shots taken and *y* the number of shots made, $\frac{y}{x} \neq 0.78$, but instead, $0.775 \leq \frac{y}{x} < 0.785$ and $0.755 \leq \frac{y+1}{x+2} < 0.765$.

We have a system of four linear inequalities.

1. $y_1 \geq 0.775x$

2. $y_2 < 0.785x$

3. $y_3 \geq 0.755x + 0.51$

4. $y_4 < 0.765x + 0.53$

Viewing the Solution Region

These inequalities create a region in the *xy*-plane. Use your calculator to graph the region. In what window should you look? Students can sketch by hand a reasonable graph for this region.

Figure 1 at right illustrates the situation. Every point in this region satisfies the four inequalities. In this region, are there any points whose coordinates are both integers? Such points are called lattice points. We need to search this region for lattice points. How can this be done? Just where in the plane is this region?

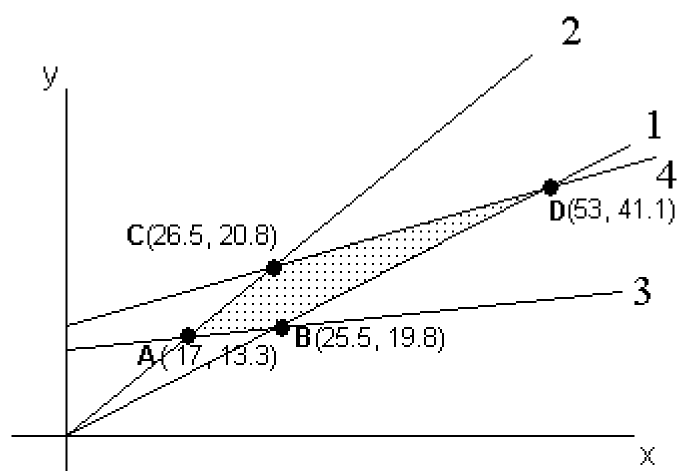

Figure 1: Hand Sketch of the Region of Interest

It is a simple matter to find the intersections of the linear boundaries indicated in Figure 1. Most students think they can just graph the equations on their calculator and trace through the region. Unfortunately, the region is so long and thin, that this is not a reasonable option. Figure 2 shows the region on the window $x \in [15, 55]$ and $y \in [12, 44]$. This is an example of a situation in

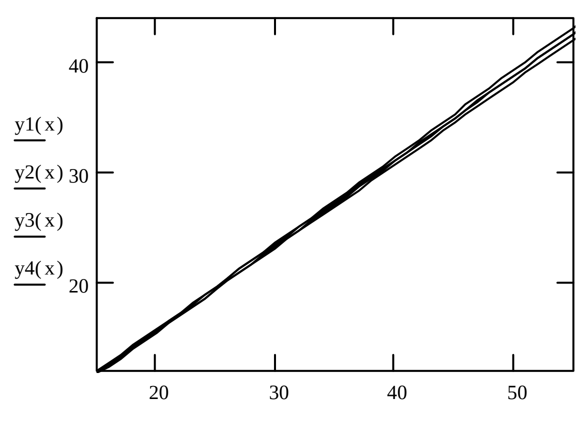

Figure 2: Computer Graph of the Region of Interest

which a hand sketch is better than a computer generated graph. There are many ways to search the region for lattice points. Here is one of them.

Check the Boundaries in Pairs

Students will recognize that there are three distinct regions in Figure 1. The region with $x \in [17,25]$ has equations 2 and 3 as upper and lower boundaries, respectively. The region with $x \in [27, 53]$ has equations 4 and 1 as upper and lower boundaries, respectively. The region between $x = 25.5$ and $x = 26.5$ has lines 4 and 3 as upper and lower boundaries, respectively. This last region is the easiest to consider. There is only one integer value of x in the region, namely, $x = 26$. On the lower boundary, the value of $y_3(26) = 20.14$ and on the upper boundary the value of $y_4(26) = 20.42$. Since there is no integer between 20.14 and 20.42, there is no integer value of *y* corresponding to $x = 26$. We need to look elsewhere for solutions.

In the region from $x = 17$ to $x = 25$, the boundaries are $y_2(x) = 0.785x$ and $y_3(x) = 0.755x + 0.51$. If we evaluate each boundary at integer values of *x*, we create Table 1 below:

x	17	18	19	20	21	22	23	24	25
$y_2(x)$	13.3	14.07	14.84	15.62	16.39	17.16	17.93	18.71	19.48
$y_3(x)$	13.35	14.13	14.92	15.70	16.49	17.27	18.06	18.84	19.63

Table 1: Evaluating on the Boundaries

Notice that when $x = 23$, the lower boundary is 17.93 and the upper boundary is 18.06. Thus, $y = 18$ lies in the region, and $x = 23, y = 18$ will satisfy the conditions of the problem with $\frac{y}{x} = \frac{18}{23} = 0.7826 \approx 0.78$ and $\frac{y+1}{x+2} = \frac{19}{25} = 0.76$. Are there other solutions?

Continuing in the portion of the region with $x \in [27, 53]$ and bounded by has equations $y_1 = 0.775x$ and $y_4 = 0.765x + 0.53$, we create Table 2.

x	27	28	29	...	32	33	...	36	37
$y_1(x)$	20.93	21.7	22.48	...	24.8	25.58	...	27.9	28.68
$y_4(x)$	21.19	21.95	22.72	...	25.01	25.78	...	28.07	28.84

x	38	39	40	41	...	49	50	...	53
$y_1(x)$	29.45	30.23	31	31.78	...	37.98	38.75	...	41.08
$y_4(x)$	29.60	30.37	31.13	31.90	...	38.02	38.78	...	41.08

Table 2: Evaluating Along the Boundaries

From Table 2, we see there are 5 additional solutions. Notice that 31 lies exactly on line y_1. Since this is a boundary that allows equality, we can keep that solution. Altogether, there are six lattice points in the region shown in Figure 2. The six solutions are:

- 18 successes in 23 attempts gives $\frac{y}{x} = \frac{18}{23} = 0.7826$ and $\frac{y+1}{x+2} = \frac{19}{25} = 0.76$.
- 21 successes in 27 attempts gives $\frac{y}{x} = \frac{21}{27} = 0.7778$ and $\frac{y+1}{x+2} = \frac{22}{29} = 0.7586$.
- 25 successes in 32 attempts gives $\frac{y}{x} = \frac{25}{32} = 0.7813$ and $\frac{y+1}{x+2} = \frac{26}{43} = 0.7647$.
- 28 successes in 36 attempts gives $\frac{y}{x} = \frac{28}{36} = 0.7778$ and $\frac{y+1}{x+2} = \frac{29}{38} = 0.7632$.
- 31 successes in 40 attempts gives $\frac{y}{x} = \frac{31}{40} = 0.775$ and $\frac{y+1}{x+2} = \frac{32}{42} = 0.7619$.
- 38 successes in 49 attempts gives $\frac{y}{x} = \frac{38}{49} = 0.7755$ and $\frac{y+1}{x+2} = \frac{39}{51} = 0.7647$.

So, there are six solutions to the problem.