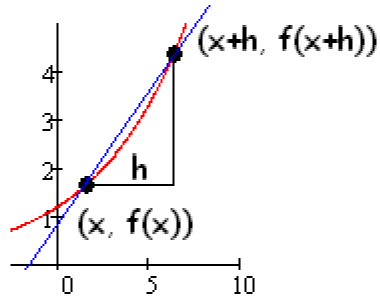
Calculus Challenge #2 Solution

Classic definition of Derivative:

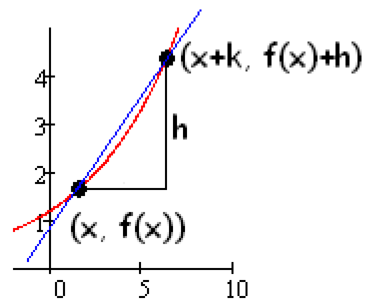
It is traditional to define the derivative as $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. The secant line geometry of this suggests that we pick a horizontal distance h, take whatever change in y we need to create the secant line and compute the slope. We then shrink h to zero and consider the limiting value of the slope.



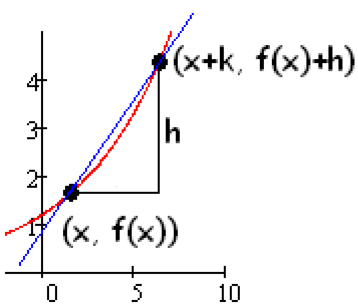
The "Modified" Derivative:

Why not do it the other way around? Why not fix a vertical distance h, take whatever change in x we need to create a secant line and compute the slope. Then shrink h to zero and consider the limiting value as before.

What happens if you do it this way?



a)Show that the value of k in the diagram above is $k = f^{-1}\left(f(x)+h\right)-x$.



The diagram at left illustrates the situation. We have $f(x+k) = f(x)+h$. Solving for k, we find that $x+k = f^{-1}\left(f(x)+h\right)$, so $k = f^{-1}\left(f(x)+h\right)-x$ as required.

b) This gives a new definition of derivative, $\lim_{h\to 0}\frac{h}{f^{-1}\left(f(x)+h\right)-x}$. Use this definition to find the derivative of :

i) $f(x)=\sqrt{x}$.

When $f(x)=\sqrt{x}$, the definition $\lim_{h\to 0}\frac{h}{f^{-1}\left(f(x)+h\right)-x}$ can be written as

$\lim_{h\to 0}\frac{h}{f^{-1}\left(\sqrt{x}+h\right)-x}=\lim_{h\to 0}\frac{h}{\left(\sqrt{x}+h\right)^2-x}$, since $f^{-1}(x)=x^2,\ x\geq 0$. Expanding the binomial gives us

$\lim_{h\to 0}\frac{h}{\left(x+2h\sqrt{x}+h^2\right)-x}$ which simplifies to $\lim_{h\to 0}\frac{h}{2h\sqrt{x}+h^2}$. Finally, we divide out the common factor of h to produce $\lim_{h\to 0}\frac{1}{2\sqrt{x}+h}$. At this point, the limit is easily evaluated as $\lim_{h\to 0}\frac{1}{2\sqrt{x}+h}=\frac{1}{2\sqrt{x}}$.

ii) $f(x)=\frac{1}{x}$. When $f(x)=\frac{1}{x}$, the definition $\lim_{h\to 0}\frac{h}{f^{-1}\left(f(x)+h\right)-x}$ can be written as $\lim_{h\to 0}\frac{h}{f^{-1}\left(\frac{1}{x}+h\right)-x}$. Since the reciprocal function is its own inverse, we have $\lim_{h\to 0}\frac{h}{\frac{1}{\left(\frac{1}{x}+h\right)}-x}$.

Simplifying, we have $\lim_{h\to 0}\frac{h}{\frac{1}{\left(\frac{1}{x}+h\right)}-x}=\lim_{h\to 0}\frac{h}{\frac{1}{\left(\frac{1+hx}{x}\right)}-x}$.

Simplifying the complex fraction gives $\lim_{h\to 0}\frac{h}{\frac{x}{1+hx}-x}=\lim_{h\to 0}\frac{h}{\frac{x}{1+hx}-\frac{x(1+hx)}{1+hx}}=\lim_{h\to 0}\frac{h}{\left(\frac{x-x-hx^2}{1+hx}\right)}$.

Cleaning up this result produces $\lim_{h\to 0}\frac{h}{\left(\frac{-hx^2}{1+hx}\right)}=\lim_{h\to 0}h\cdot\left(\frac{1+hx}{-hx^2}\right)$, and dividing out the *h*'s gives $\lim_{h\to 0}\left(\frac{1+hx}{-x^2}\right)=-\frac{1}{x^2}$.

iii) What difficulties arise with $f(x)=x^2$? Does the new definition work for all *x*?

Proceeding as before, we have $\lim_{h\to 0}\frac{h}{f^{-1}\left(f(x)+h\right)-x}=\lim_{h\to 0}\frac{h}{\sqrt{\left(x^2+h\right)}-x}$

We can simplify this expression by rationalizing the denominator,

$$\lim_{h\to 0}\frac{h}{\sqrt{\left(x^2+h\right)}-x}\left(\frac{\sqrt{\left(x^2+h\right)}+x}{\sqrt{\left(x^2+h\right)}+x}\right)=\lim_{h\to 0}\frac{h\left(\sqrt{\left(x^2+h\right)}+x\right)}{\left(x^2+h\right)-x^2}.$$

This in turn can be simplified to

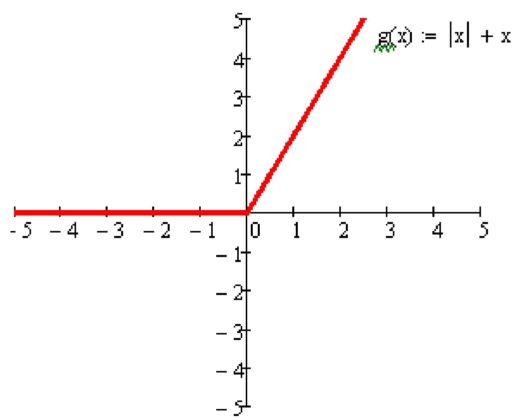
$\lim_{h \to 0} \frac{h\left(\sqrt{(x^2+h)}+x\right)}{h} = \lim_{h \to 0} \sqrt{(x^2+h)}+x$ which can now be evaluated:

$$\frac{d}{dx}\left(x^2\right) = \lim_{h \to 0} \sqrt{(x^2+h)}+x = \sqrt{x^2}+x.$$

Recall that $\sqrt{x^2} = |x|$. This gives $\frac{d}{dx}\left(x^2\right) = |x| + x$.

Now we see a problem, since $|x| = \begin{cases} x \text{ if } x \ge 0 \\ -x \text{ if } x < 0 \end{cases}$, so $|x| + x = \begin{cases} 2x \text{ if } x \ge 0 \\ 0 \text{ if } x < 0 \end{cases}$.

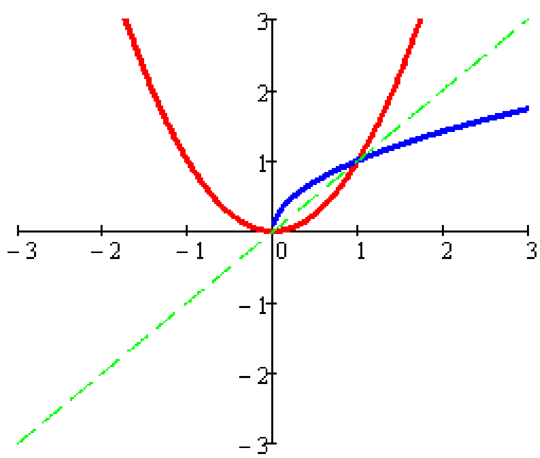
The graph of $y = |x| + x$ is shown at right. Notice that the derivative is equal to 2*x* (as expected) only for non-negative values of *x*.



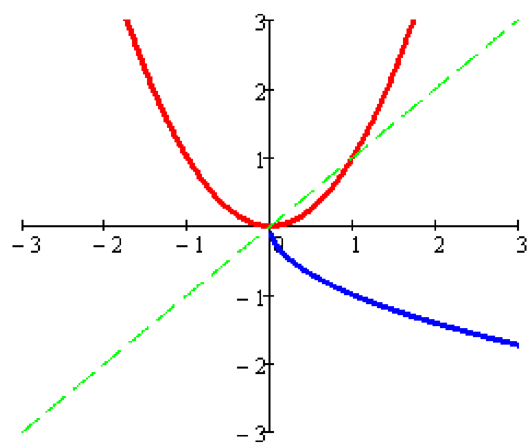
What is the problem?

Since $f(x) = x^2$ is not a one-to-one function, its inverse must be carefully considered.

If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$ only if $x \ge 0$.



The graph of $g(x) = \sqrt{x}$ is the reflection of $f(x) = x^2$ about $y = x$ only for $x \ge 0$.



The graph of $h(x) = -\sqrt{x}$ is the reflection of $f(x) = x^2$ about $y = x$ when $x < 0$.

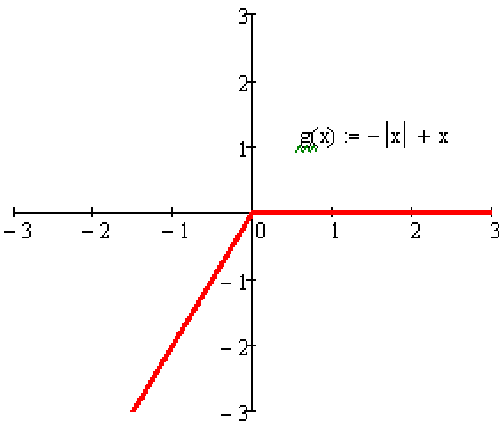
So, we need to pick up the other branch of the parabola.

So, we should have written $\lim_{h\to 0}\frac{h}{f^{-1}\left(f\left(x\right)+h\right)-x}=\lim_{h\to 0}\frac{h}{\sqrt{\left(x^2+h\right)}-x}$ only when $x\geq 0$.

So, repeating the algebra above, but with the caveat $x\geq 0$, we find that $\lim_{h\to 0}\frac{h}{\sqrt{\left(x^2+h\right)}-x}=\left|x\right|+x$, for $x\geq 0$. So, $\frac{d}{dx}\left(x^2\right)=\left|x\right|+x=2x$, when $x\geq 0$.

However, if $x<0$, we have $\lim_{h\to 0}\frac{h}{f^{-1}\left(f\left(x\right)+h\right)-x}=\lim_{h\to 0}\frac{h}{-\sqrt{\left(x^2+h\right)}-x}$ when $x<0$. Notice the negative sign in front of the square root.

As before, we simplify so that $\lim_{h\to 0}\frac{h}{-\sqrt{\left(x^2+h\right)}-x}=-\sqrt{x^2}+x=-\left|x\right|+x$ when $x<0$. But, if $x<0$, then $\left|x\right|=-x$, so when $x<0$, $-\left|x\right|=x$. This gives, $\frac{d}{dx}\left(x^2\right)=-\left|x\right|+x=2x$ if $x<0$. The graph is shown at right.



The final consideration is for $x=0$. In this case we have $\lim_{h\to 0}\frac{h}{\sqrt{\left(0^2+h\right)}-0}=\lim_{h\to 0}\sqrt{h}$. The limit from the right exists and is equal to zero, but the limit from the left does not exist. Consequently, this limit does not exist. So we have a problem with our new definition at $x=0$.

So, we do have $\frac{d}{dx}\left(x^2\right)=2x$ for all $x\neq 0$. This derivative will have trouble whenever the original function is not one-to-one. You should find it easier to use for functions like $f\left(x\right)=\sqrt[3]{x}$, but not very useful at all for trigonometric or exponential functions.