Calculus Challenge Problem #3 SOLUTION

1. Extend the product rule to more than two differentiable functions.

a) $\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\cdot f_3(x)\right)$

b) $\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\cdot f_3(x)\cdot f_4(x)\right)$

c) Find the derivative for a product of *k* differentiable functions $\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\cdots f_k(x)\right)$

These derivatives are pretty straightforward. By grouping two of the functions, we can use the product rule.

$$\frac{d}{dx}\left(f_1\cdot f_2\cdot f_3\right)=\frac{d}{dx}\left(\left(f_1\cdot f_2\right)\cdot f_3\right)=\left(f_1\cdot f_2\right)\cdot f_3'+\left(f_1\cdot f_2\right)'\cdot f_3$$

$$=\left(f_1\cdot f_2\right)\cdot f_3'+\left(f_1'\cdot f_2+f_1\cdot f_2'\right)'\cdot f_3=f_1\cdot f_2\cdot f_3'+f_1\cdot f_2'\cdot f_3+f_1'\cdot f_2\cdot f_3$$

Repeating the process using the result we just derived.

$$\frac{d}{dx}\left(f_1\cdot f_2\cdot f_3\cdot f_4\right)=\frac{d}{dx}\left(\left(f_1\cdot f_2\cdot f_3\right)\cdot f_4\right)=\left(f_1\cdot f_2\cdot f_3\right)\cdot f_4'+\left(f_1\cdot f_2\cdot f_3\right)'\cdot f_4$$

$$=\left(f_1\cdot f_2\cdot f_3\right)\cdot f_4'+\left(f_1\cdot f_2\cdot f_3'+f_1\cdot f_2'\cdot f_3+f_1'\cdot f_2\cdot f_3\right)\cdot f_4$$

$$=f_1\cdot f_2\cdot f_3\cdot f_4'+f_1\cdot f_2\cdot f_3'\cdot f_4+f_1\cdot f_2'\cdot f_3\cdot f_4+f_1'\cdot f_2\cdot f_3\cdot f_4$$

The derivative of five functions can use this result. We see that the product rule for *k* differentiable functions will be the product of the *k* functions with one of them differentiated.

So,

$$\frac{d}{dx}\left(f_1\cdot f_2\cdots\cdot f_k\right)=f_1\cdot f_2\cdots\cdot f_{k-1}\cdot f_k'+f_1\cdot f_2\cdots\cdot f_{k-1}'\cdot f_k+\cdots+f_1\cdot f_2'\cdots f_{k-1}'\cdot f_k+f_1'\cdot f_2\cdots f_{k-1}'\cdot f_k$$

2. Find a product rule for:

a) the second derivative $\dfrac{d^2}{dx^2}\left(f(x)\cdot g(x)\right)$.

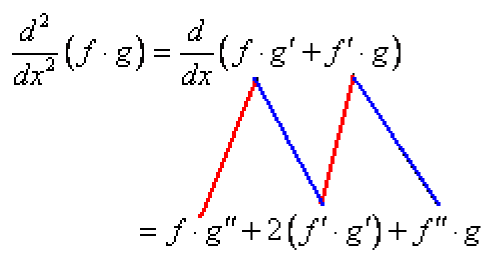
b) the third derivative $\dfrac{d^3}{dx^3}\left(f(x)\cdot g(x)\right)$.

c) Look at the 4th and 5th derivatives to generalize to the n^{th} derivative $\dfrac{d^n}{dx^n}\left(f(x)\cdot g(x)\right)$. Explain the pattern that you see in the coefficient

Now things get a little more interesting…

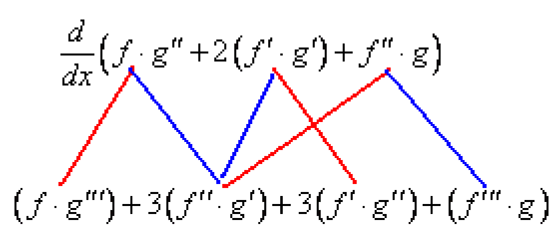
$$\frac{d^2}{dx^2}(f\cdot g)=\frac{d}{dx}(f\cdot g'+f'\cdot g)=(f\cdot g''+f'\cdot g')+(f'\cdot g'+f\cdot g'')$$
$$=f\cdot g''+2(f'\cdot g')+f''\cdot g$$

Notice that in each product, we generate two new terms. One containing the derivative of the *g* function on one higher order (shown in red) and the other the derivative of the *f* function of one higher order (shown in blue). This explains the patterns we see.



$$\frac{d^3}{dx^3}(f\cdot g)=\frac{d}{dx}\left(f\cdot g''+2(f'\cdot g')+f''\cdot g\right)=(f\cdot g'''+f'\cdot g'')+2(f'\cdot g''+f''\cdot g')(f''\cdot g'+f'''\cdot g)$$
$$=(f\cdot g''')+3(f''\cdot g')+3(f'\cdot g'')+(f'''\cdot g)$$

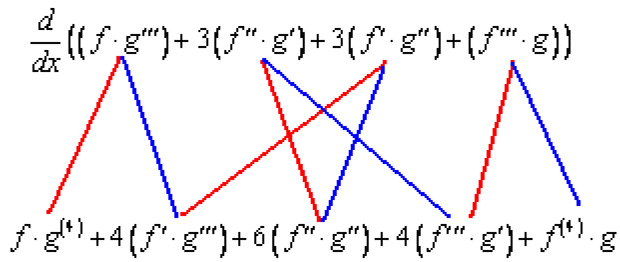
The pattern continues. Since there are 2 of the $f'\cdot g'$ terms generating an $f''\cdot g'$ and one $f''\cdot g$ contributing, we have a total of three $f''\cdot g'$ terms. The new total is the sum of the contributing components.



$$\frac{d^4}{dx^4}(f\cdot g)=\frac{d}{dx}\left((f\cdot g''')+3(f''\cdot g')+3(f'\cdot g'')+(f'''\cdot g)\right).$$

This derivative will have the following terms: $f\cdot g^{(4)},\ f'\cdot g''',\ f''\cdot g'',\ f'''\cdot g',$ and $f^{(4)}\cdot g$.

The term $f\cdot g^{(4)}$ can only come from differentiating $f\cdot g'''$, so we will have one of them.



The term $f'\cdot g'''$ comes from the previous $f\cdot g'''$ and $f'\cdot g''$ terms and adding them together.
This give us 1 + 3 = 4 of these terms.

The term $f''\cdot g''$ comes from differentiating $f'\cdot g''$ and from $f''\cdot g'$ and adding them together.
This give us 3 + 3 = 6 of these terms.

The other terms are creating similarly.

So, we have $\frac{d^4}{dx^4}(f\cdot g)=f\cdot g^{(4)}+4(f'\cdot g''')+6(f''\cdot g'')+4(f'''\cdot g')+f^{(4)}\cdot g$

We see the terms in Pascal's triangle appearing as the coefficients.

In general, $\frac{d^n}{dx^n}(f\cdot g)=\binom{n}{0}f\cdot g^{(n)}+\binom{n}{1}f'\cdot g^{(n-1)}+\binom{n}{2}f''\cdot g^{(n-2)}+\cdots\binom{n}{n}f^{(n)}\cdot g$.

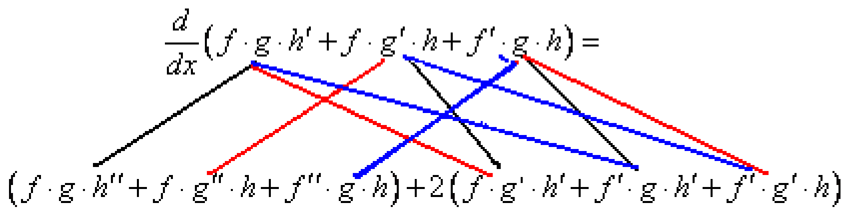
Also notice that the coefficient of $f^{(m)}\cdot g^{(m)}$ is $\binom{n+m}{n}$ or $\frac{(n+m)!}{n!\,m!}$. This will be important in the next section.

3. Combine these ideas to find $\frac{d^2}{dx^2}(f\cdot g\cdot h)$ and $\frac{d^3}{dx^3}(f\cdot g\cdot h)$. Explain the pattern you see in the coefficients.

We know from part 1) that $\frac{d}{dx}(f\cdot g\cdot h)=f\cdot g\cdot h'+f\cdot g'\cdot h+f'\cdot g\cdot h$, so

$$\frac{d}{dx}(f\cdot g\cdot h'+f\cdot g'\cdot h+f'\cdot g\cdot h)=$$
$$(f\cdot g\cdot h''+f\cdot g'\cdot h'+f'\cdot g\cdot h')+(f\cdot g'\cdot h'+f\cdot g''\cdot h+f'\cdot g'\cdot h)+(f'\cdot g\cdot h'+f'\cdot g'\cdot h+f''\cdot g\cdot h)=$$
$$(f\cdot g\cdot h''+f\cdot g''\cdot h+f''\cdot g\cdot h)+2(f\cdot g'\cdot h'+f'\cdot g\cdot h'+f'\cdot g'\cdot h)$$

The diagram is getting a little messy to try to draw, but the same combinations of terms happens.



Repeating with $\frac{d^3}{dx^3}\left(f\cdot g\cdot h\right)$, we see that each term in $\left(f\cdot g\cdot h''+f\cdot g''\cdot h+f''\cdot g\cdot h\right)+2\left(f\cdot g'\cdot h'+f'\cdot g\cdot h'+f'\cdot g'\cdot h\right)$ generates three new components that may add together. We will have all possible partitions of 3 as the orders of the derivatives. That is we will have (3,0,0) in the form of $f'''\cdot g\cdot h$ and similar terms, (2, 1,0) in the form of $f\cdot g''\cdot h'$ and similar terms, and (1, 1, 1) in the form of $f'\cdot g'\cdot h'$.

We just need to count how many of each we have and to do that, we look at where they come from in $\left(f\cdot g\cdot h''+f\cdot g''\cdot h+f''\cdot g\cdot h\right)+2\left(f\cdot g'\cdot h'+f'\cdot g\cdot h'+f'\cdot g'\cdot h\right)$.

So, we find that

$$\frac{d^3}{dx^3}\left(f\cdot g\cdot h\right)=\left(f\cdot g\cdot h'''+f\cdot g'''\cdot h+f'''\cdot g\cdot h\right)$$
$$+3\left(f\cdot g'\cdot h''+f\cdot g''\cdot h'+f'\cdot g\cdot h''+f'\cdot g''\cdot h+f''\cdot g\cdot h'+f''\cdot g'\cdot h\right)$$
$$+6\left(f'\cdot g'\cdot h'\right)$$

How does the pattern seen earlier fit (the coefficient of $f^{(m)}\cdot g^{(m)}$ is $\frac{(n+m)!}{n!m!}$). Is the coefficient of $f^{(n)}\cdot g^{(m)}\cdot h^{(k)}$ equal to $\frac{(n+m+k)!}{n!m!k!}$? We have $\frac{(3+0+0)!}{3!0!0!}=1$, $\frac{(2+1+0)!}{2!1!0!}=3$, and $\frac{(1+1+1)!}{1!1!1!}=6$. Nice. These coefficients are known as multinomial coefficients.

Let’s try one more.

$\frac{d^4}{dx^4}\left(f\cdot g\cdot h\right)$ should have terms with derivatives of orders (4, 0, 0), (3, 1, 0), (2, 2, 0), (2, 1, 1).

The coefficients should be $\frac{(4+0+0)!}{4!0!0!}=1$, $\frac{(3+1+0)!}{3!1!0!}=4$, $\frac{(2+2+0)!}{2!2!0!}=6$, and $\frac{(2+1+1)!}{2!1!1!}=12$.

If we crank out the derivatives and add like terms, we see this is correct.

4. Suppose I find $\frac{d^5}{dx^5}\left(f \cdot g \cdot h \cdot k\right)$. Is there a $f''g\,h'k''$ term in the expansion? If there is, what is its coefficient? If there isn't, explain why this term does not exist.

From our earlier work, we see that the terms of $\frac{d^n}{dx^n}\left(f \cdot g \cdot h \cdot k\right)$ are

$\frac{n!}{(n_1!)(n_2!)(n_2!)(n_2!)} f^{(n_1)} g^{(n_2)} h^{(n_3)} k^{(n_4)}$ where $n_1 + n_2 + n_3 + n_4 = n$.

There will be a $f''g\,h'k''$ term in the expansion and every other partition of 5 into 4 groups as orders of the derivatives (there will not be terms like $f'g\,h'k'$ or $f''g'\,h'k''$, where the orders do not add to 5).

So, the coefficient of $f'' \cdot g \cdot h' \cdot k''$ is $\frac{5!}{(2!)(0!)(1!)(2!)} = \frac{120}{4} = 30$.